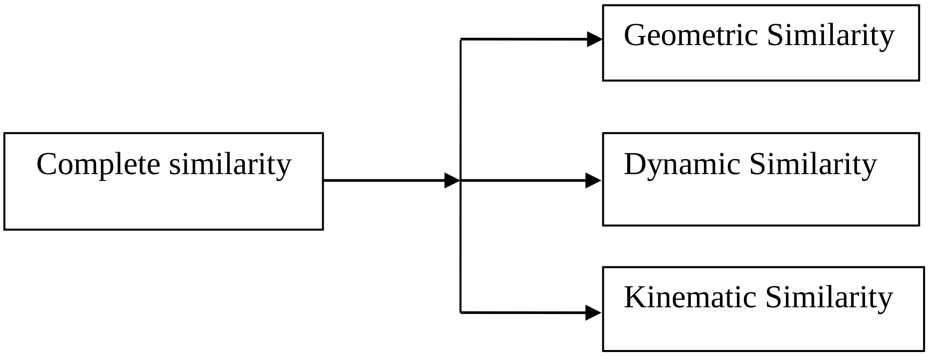
CHAPTER 6

Dynamic Similarity and Dimensional Analysis

6.1 *Definition of Physical Similarity.*

Two systems described by the same physics operating under different set of conditions are said to be physically similar in respect of certain specified physical quantities, when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

There are three types of similarities as in following chart which constitute the complete similarity between problems of same kind.



6.2 Geometric Similarity (G.S).

Geometric similarity implies the similarity of shape such that, the ratio of any length in one system to the corresponding length in other system is the same everywhere.

- Prototype:- is the full size or actual scale systems.
- Models:- is the laboratory scale systems.
- The model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity and properties of the fluid.

$$length\ ratio = \frac{length\ of\ model}{length\ of\ prototype}$$

Or $L_r = \frac{L_m}{L_p}$

$$area\ ratio = \frac{model\ area}{prototype\ area} = \frac{L_m^2}{L_p^2}$$

$$L^2 = L_r^2$$

L_r Known as the model ratio or is the scale factor

6.3 Kinematic Similarity (K.S).

Kinematic similarity refers to similarity of motion

$$\text{Motion} \longrightarrow \begin{cases} \text{Distance} \longrightarrow \text{Similarity of length (i.e., G.S.)} \\ \text{Time} \longrightarrow \text{Similarity of time intervals.} \end{cases}$$

$$velocity\ ratio = \frac{V_m}{V_p} = \frac{L_m/T_m}{L_p/T_p} = \frac{L_m}{L_p} \div \frac{T_m}{T_p} = \frac{L_r}{T_r}$$

$$accelartion\ ratio = \frac{L_m/{T_m}^2}{L_p/{T_p}^2} = \frac{L_m}{L_p} \div \frac{T_m^2}{T_p^2} = \frac{L_r}{T_r^2}$$

$$flow\ rate\ ratio = \frac{Q_m}{Q_p} = \frac{{L_m}^3/T_m}{{L_p}^3/T_p} = \frac{L_m^3}{L_P^3} \div \frac{T_m}{T_p} = \frac{L_r^3}{T_r}$$

Therefore, geometric similarity is a necessary condition for the kinematic similarity to be achieved, but not the sufficient one.

6.4 Dynamic Similarity (D.S).

Dynamic similarity is the similarity of forces, in dynamically similar system, the magnitudes of forces at correspondingly similar point in each system are in a fixed ratio. In a system involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows;

- Viscous force (due to viscosity) $\vec{F_v}$
- Pressure force (due to different in pressure) $\vec{F_p}$
- Gravity force (due to gravitational attraction) $\vec{F_g}$
- Capillary force (due to surface tension) $\vec{F_c}$
- Compressibility force (due to elasticity) $\vec{F_e}$

According to Newton's law, the resultant F_R of all these forces, will cause the accelartion of a fluid element, hence

$$\vec{F}_R = \vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e \qquad (6.1)$$

The inertia force $\vec{F}_i$ is defined as equal and opposite to the resultant accelerating force $\vec{F}_R$

$$\vec{F}_i = -\vec{F}_R$$

$\therefore Eq. (6.1)\ can\ be\ expresed\ as$

$$\vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e + \vec{F}_i = 0$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both prototype and the model. The inertia force $\vec{F}_i$ is usually taken as the common one to describe the ratio

$$\frac{|\vec{F}_v|}{|\vec{F}_i|}, \frac{|\vec{F}_p|}{|\vec{F}_i|}, \frac{|\vec{F}_g|}{|\vec{F}_i|}, \frac{|\vec{F}_c|}{|\vec{F}_i|}, \frac{|\vec{F}_e|}{|\vec{F}_i|}$$

a- <u>Inertia Force</u>.

The inertia force is the force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.

Mass of element $\propto \rho L^3$, ρ is the density, L is the characteristic length, acceleration of a fluid element is the rate change of velocity in that direction change with time

$$a \propto \frac{V}{t}\ ; \quad t \propto \frac{L}{V}$$

$$\therefore\ a \propto \frac{V^2}{L}$$

The magnitude of inertia force is thus proportional to $\rho\, L^3 \frac{V^2}{L} = \rho\, L^2 V^2$

This can be written as $\left|\vec{F}_i\right| \propto \rho L^2 V^2$ (6.2)

b- <u>Viscous Force.</u>

The viscous force arises from shear stress in a flow of fluid, therefor, we can write magnitude of viscous force $\vec{F}_v$= shear stress * surface area.

Shear stress = μ(viscosity) * rate of shear strain

Where rate of shear strain ∝ velocity gradient $\propto \frac{V}{L}$

Surface area $\propto L^2$

$$\left|\mathrm{F_v}\right| \propto \mu \frac{\mathrm{V}}{\mathrm{L}} \mathrm{L}^2 \propto \mu \mathrm{VL} \qquad (6.3)$$

c- <u>Pressure Force.</u>

The pressure force arises due to the difference of pressure in a flow field. Hence it can be written as

$$\left|\vec{F}_P\right| \propto \Delta P L^2 \qquad (6.4)$$

Where Δp is some characteristic pressure in the flow.

d- Gravity Force.

The gravity force on a fluid element is its weight, hence,

$$\left|\vec{F}_g\right| \propto \rho L^3 g \qquad (6.5)$$

Where g is the acceleration due to gravity (or weight per unit mass)

e- Capillary or Surface Tension Force.

The capillary force arises due to the existence of an interface between two fluids. It is equal to the coefficient of surface tension σ multiplied by the length of a linear element on the surface perpendicular to which the force acts, therefore,

$$\left|\vec{F}_c\right| \propto \sigma l \qquad (6.6)$$

f- Compressibility or Elastic Force.

For a given compression (a decrease in volume), the increase in pressure is proportional to the bulk modulus of elasticity E $(\Delta p \propto E)$, this gives rise to a force known as the elastic force .

$$\left|\vec{F}_c\right| \propto E L^2 \qquad (6.7)$$

Note, the flow of fluid in practice does not involve all the forces simultaneously.

6.4.1 D. S. of Flow Governed by Viscous, Pressure and Inertia Forces.

The ratios of the representative magnitudes of these forces with the help of Eq's (4.2) to (4.5) as follows:

$$\frac{Viscousforce}{Inertiaforce} = \frac{|\vec{F}_v|}{|\vec{F}_i|} \propto \frac{\mu V L}{\rho V^2 L^2} = \frac{\mu}{\rho V L} \qquad (6.8)$$

$$\frac{pressureforce}{Inertiaforce} = \frac{|\vec{F}_p|}{|\vec{F}_i|} \propto \frac{\Delta p L^2}{\rho V^2 L^2} = \frac{\Delta p}{\rho V^2} \qquad (6.9)$$

The term ρLV/μ is known as Reynolds number, ***Re***.

$Re \propto \frac{Inertia\ force}{Viscous\ force}$ is thus proportion to the magnitude ratio.

The term $\frac{\Delta p}{\rho V^2}$ is known as Euler number, $\boldsymbol{Eu}$.

$\therefore$ $\boldsymbol{Re}$ & $\boldsymbol{Eu}$ Represent the criteria of ***D.S.*** for the flows which are affected only by viscous, pressure and iertia forces. For example are

1- The full flow of fluid in a completely closed conduit
2- Flow of air past a low – speed aircraft
3- The flow of water past a submarine deeply submerged to produce no waves on the surface.

Hence, ***Re & Eu*** for a complete dynamic similarity between prototype and model must be the same for two. Thus

$$\frac{\rho_p\, L_p V_p}{\mu_p} = \frac{\rho_m\, L_m V_m}{\mu_m} \qquad (6.10)$$

$$\frac{\Delta p_p}{\rho_p V_p^2} = \frac{\Delta p_m}{\rho_m\, V_m^2} \qquad (6.11)$$

6.4.2 D.S. of Flow Governed by Gravity and Inertia Forces.

A flow of the type in which significant force are gravity and inertia forces, is found when a free surface is present. For example are

1- The flow of a liquid in an open channel.
2- The wave motion caused by the passage of a ship through water.
3- The flow over weirs and spillways.

The condition for ***D.S.*** of such flows requires

- The equality of ***Eu.***
- The equality of the magnitude ratio of gravity to inertia force at corresponding points in the system.

$$\frac{Gravityforce}{Inertiaforce} = \frac{|\vec{F}_g|}{|\vec{F}_i|} \propto \frac{\rho L^3 g}{\rho V^2 L^2} = \frac{Lg}{V^2} \qquad (6.12)$$

The reciprocal the term $\frac{(Lg)^{\frac{1}{2}}}{V}$ is known as Froude number, ***Fr***

$$\therefore\ Fr = \frac{V}{(Lg)^{\frac{1}{2}}}$$

$\therefore$ Dynamic similarity between prototype & model is the equality of Froude number

$$\frac{\sqrt{L_p\, g_p}}{V_p} = \frac{\sqrt{L_m\, g_m}}{V_m} \qquad (6.13)$$

6.4.3 D.S. of Flows with Surface Tension as the Dominant Force.

Surface tension forces are important in certain classes of practical problems such as:

1- Flows in which capillary waves appear.
2- Flows of small jets and thin sheets of liquid injected by nozzle in air.
3- Flow of a thin sheet of liquid over a solid surface.

Dynamic similarity is the magnitude ratio

$$\frac{|\vec{F}_c|}{|\vec{F}_i|} \propto \frac{\sigma L}{\rho V^2 L^2} = \frac{\sigma}{\rho V^2 L} \qquad (6.14)$$

The term $\frac{\sigma}{\rho V^2 L}$ is usually knows as Weber number, $\boldsymbol{Wb}$.

For dynamic similarity $(Wb\ )_m = (Wb\ )_p$

$$i.e., \frac{\sigma_m}{\rho_m V_m^2 L_m} = \frac{\sigma_p}{\rho_p V_p^2 L_p} \qquad (6.15)$$

6.4.4 D.S. of Flows with Elastic Force.

The magnitude ratio of inertia to elastic force becomes

$$\frac{\text{Inertia force}}{\text{Elastic force}} = \frac{|\vec{F}_i|}{|\vec{F}_e|} \propto \frac{\rho V^2 L^2}{E L^2} = \frac{\rho V^2}{E} \qquad (6.16)$$

The parameter $\frac{\rho V^2}{E}$ is known as Cauchy number.

For dynamic similarity flow ***(Cauchy)m = (Cauchy)p***

$$i.e., \frac{\rho_m V_m^2}{(E_s)m} = \frac{\rho_p V_p^2}{(E_s)_P} \qquad (6.17)$$

If the flow is isentropic $E = E_s$ is isentropic bulk modulus of elasticity.

i = sound wave propagates through a fluid medium = $\sqrt{\frac{E_s}{\rho}}$

$\therefore$ $the\ term\ \rho V^2/E_s$ can be written as V^2/i^2

The ratio $\left(\frac{V}{i}\right) = \boldsymbol{Ma}$ is known as ***Mach number***, in the flow of air past high-speed aircraft, missiles, propellers and rotary compressors. In these cases equality of Mach number is a condition of dynamic similarity. Therefore

$(Ma\ )_p = (Ma\ )_m$

$$i.e \left(\frac{V_p}{i_p}\right) = \left(\frac{V_m}{i_m}\right) \qquad (6.18)$$

Dimensionless terms	Representation magnitude ratio of the force	Name	Recommended symbol
$\rho LV/\mu$	$\frac{\text{Inertiaforce}}{\text{viscous force}}$	Reynolds number	Re
$\Delta p/\rho V^2$	$\frac{\text{pressureforce}}{\text{Inertiaforce}}$	Euler number	Eu
$V/(Lg)^{1/2}$	$\frac{\text{Inertia force}}{\text{Gravity force}}$	Froud number	Fr
$\frac{\sigma}{\rho V^2 L}$	$\frac{\text{surface tension}}{\text{Inertia force}}$	Weber number	Wb
$V/\sqrt{E_s/\rho}$	$\frac{\text{Inertiaforce}}{\text{Elastic force}}$	Mach number	Ma

Ex.1

When tested in water at 20C° flowing at 2 m/s, an 8-cm diameter sphere has a measured drag of 5 N. What will be the velocity and drag force on a 1.5m diameter weather balloon moored in sea-level standard air under dynamically similar condition?

Sol.

For water at 20C° $\rho \approx 998\frac{kg}{m^3}, \&\ \mu = 0.001\frac{kg}{m.s}$

For air at sea level $\rho \approx 1.2255\frac{kg}{m^3}\ , \mu = 1.78 *\ 10^{-5}\ \ \frac{kg}{m.s}$

The balloon velocity follows from dynamic similarity, which requires identical (***Reynolds number***).

$$\boldsymbol{Re_m} = \boldsymbol{Re_p} = \frac{\rho VD}{\mu}$$

$$\mathrm{Re_m} = \frac{998*(2.0)(0.08)}{0.001} \approx 1.6 * 10^5 = \mathrm{Re_p}$$

$$1.6 * 10^5 = 1.2255\frac{V_{ballon}(1.5)}{1.78*10^{-5}} ----\rightarrow V_{ballon} \approx 1.55\frac{m}{s}$$

Then the two spheres will be identical drag coefficients:

$$\mathrm{C_{D,m}} = \frac{\Delta p}{\rho V^2} = \left[\frac{F}{\rho V^2\left(\frac{\pi d^2}{4}\right)}\right]_m = \left[\frac{F}{\rho V^2\left(\frac{\pi d^2}{4}\right)}\right]_p = \frac{\mathrm{F}}{\mathrm{\rho V^2 d^2}}$$

$$C_{D,m} = \frac{5}{998(2)^2(0.08)^2\frac{\pi}{4}} = 0.4986 = C_{D,p} = \frac{F_{balon}}{1.2255(1.55)^2(1.5)^2\pi/4}$$

Solve for $f_{ballon} \approx 1.296\ N$

Ex.2 A model of a reservoir having a free water surface within it is drained in 3 minutes by opening a sluice gate. The geometrical scale of the model is (1/100). How long would it take to empty the prototype?

Sol.

$$\frac{Q_m}{Q_p} = \frac{\frac{L_m^3}{T_m}}{\frac{L_p^3}{T_p}} = \frac{L_r^3}{T_r} \qquad T_r = \frac{T_m}{T_p}$$

The forces control the flow is

1- Gravity force $= \frac{(F_g)_m}{(F_g)_p} = \frac{(\rho g L^3)_m}{(\rho g L^3\ )_p} = \frac{W_m L_m^3}{W_p L_P^3} = W_r\ L_r^3$

2- Inertia force $= \frac{(F_i)_m}{(F_i)_p} = \frac{(m\quad a)_m}{(m\quad a)_p} = \frac{\rho_m}{\rho_p}\frac{L_m^3}{L_p^3}\left(\frac{L_r}{T_r^2}\right) = \rho_r\ L_r^3\frac{L_r}{T_r^2}$

By equating the two ratio

$$W_r\ L_r^3 = \rho_r\ L_r^3\frac{L_r}{T_r^2}$$

$$\rho_r\ g_r\ L_r^3 = \rho_r\ L_r^3\frac{L_r}{T_r^2}$$

$$T_r^2 = \frac{L_r}{g_r} - - - - \rightarrow \frac{T_m^2}{T_p^2} = \frac{\frac{1}{100}}{1}; \quad since\ g_r = 1 \quad T_m = 3\ min$$

$$\therefore T_p = \sqrt{900} = 30\,\mathrm{min}$$

6.5 The Application of D.S and the Dimensional Analysis.

6.5.1 The concept.

A physical problem may be characterized by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables. This gives a clue to reduction in the number of parameters requiring separate consideration in an experimental investigation.

Ex:- $Re \ = \frac{\rho V D_h}{\mu}$ $\quad Re$ 2000 ~ 4000 by varying V without change in any other independent dimensional variable .

In fact, the variation in the Re physically implies the variation in any of the dimensional parameters defining it.

6.5.2 *Dimensional Analysis.*

The dimensional analysis is a mathematical technique by which can be determining many dimensionless parameters and solving several engineering problems.

There are two existing approaches:-

1- Indicial method.
2- Buckingham's ***pi*** theorem.

The dimensional analysis can be explain by the following,

- The Various physical qumtities used in fluid phenomenon can be expressed in terms of fundamental quantities or primary quantities.
- Fundamental quantities are Mass (M), Length (L), Time (T), Temperature (θ) is used for compressible flow.
- The quantities which are expressed in terms of the fundamental or primary quantities are called derived or secondary quantities as (velocity, area, acceleration)
- The expression for a derived quantities in terms of the primary quantities is called the dimension of the physically quantities.
- A quantity may either be expressed dimensionally in M-L-T or F-L-T system.

Ex.3

Determine the dimensions of the following quantities.

(i) Discharge.
(ii) Kinematic viscosity.
(iii) Force.
(iv) Specific weight.

Sol.

(i) Discharge = area * velocity

$$= L^2 * \frac{L}{T} = \frac{L^3}{T} = L^3 T^{-1}$$

(ii) Kinematic Viscosity(υ)=μ/ρ

Where (μ) given by (τ) = $\mu \frac{du}{dy}$

$$\mu = \frac{\tau}{du/_{dy}} = \frac{shearstress}{\frac{L}{T}\times\frac{1}{L}} = \frac{\frac{force}{Area}}{\frac{1}{T}}$$

$$\mu = \frac{mass\times acceleration}{Area\times\frac{1}{T}} = \frac{M\times\frac{L}{T^2}}{L^2\times\frac{1}{T}} = \frac{M\times L}{L^2T^2\times\frac{1}{T}}$$

$\mu = \frac{M}{LT} = ML^{-1}T^{-1}$

$and\ \ \rho = \frac{mass}{volume} = \frac{M}{L^3} = \ ML^{-3}$

$\therefore\ \upsilon = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1}$

(iii) Force = mass * acceleration

$= M * \frac{length}{Time^2} = \frac{ML}{T^2} = MLT^{-2}$

(iv) Specific weight= Weight/volume= force/volume= $\frac{MLT^{-2}}{L^3} = ML^{-2}T^{-2}$

6.5.3 Dimensions of Physical Quantities.

All physical quantities are expressed by magnitude and units as an example;

Velocity = 8 m/s; Acceleration = 10 m/s^2

(8 & 10 are the magnitudes but m/s & m/s^2 are the dimensions)

SI(system international) units, in fluid mechanics the primary physical quantities or (base dimensions) are the [**Mass, Length, Time, Temperature**] are symbolized as [$\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{T}, \boldsymbol{\theta}$]. Any physical quantity can be expressed in terms of these primary quantities by using the basic mathematical definition of the quantity, resulting is known as the dimension of the quantity, by substitute the mass by force (F)

$F = MLT^{-2} - - - - - - \rightarrow M = FT^2L^{-1}$

Let us take some examples.

1- *Dimension of stress.*

$shear\ stress\ \tau = \frac{force}{area}, \quad force = mass * accelarion$

Dimension of acceleration = dimension of velocity/ dimension of time

$= Dim. of\ distance/(Dim. of\ time)^2 = \frac{L}{T^2}$

$Dim. of\ area\ (length)^2 = L^2$

$\tau = \frac{\frac{ML}{T^2}}{L^2} = ML^{-1}T^{-2}$

2- *Dimension of viscosity.*

$\tau = \mu \frac{du}{dy}$

$Or\ , \qquad \mu = \frac{\tau}{\frac{du}{dy}}$

The dimension of velocity gradient du/dy can be written as du/dy = dimension of u/ dimension of y= $\frac{\frac{L}{T}}{L} = T^{-1}$

Dimension of μ

$\therefore \mu = \frac{\text{Dim.of } \tau}{\text{Dim.of du/dy}} = \frac{ML^{-1}T^{-2}}{T^{-1}} = ML^{-1}T^{-1}$

H.W. Drive the dimensions of the following physical Quantities in [M,L,T], momentum, work , weight, flow rate , Power.

6.6 Rayleigh's Indicial Method (Method-1).

Based on the fundamental principle of dimensional homogeneity of physical variables.

Procedure.

1- The dependent variable is identified and expressed as a product of all the independent variables raised to an unknown integer exponent.

2- Equating the indices of (n) fundamental dimensions of the variables involved, (n) independent equations are obtained.

3- These (n) equations are solved to obtain the dimensionless groups.

Ex.4

Let us illustrate this method by solving the pipe flow problem with $\Delta p/l$ along the pipe.

Step_1. Here, the dependent variable $\Delta p/l$ can be written as

$\frac{\Delta p}{l} = K(V^a D_h^b \rho^c \mu^d)$ Where, K is constant.

Step_2. Inserting the dimension of each variable in the above equation, we obtain.

$ML^{-2}T^{-2} = K(LT^{-1})^a (L)^b (ML^{-3})^c (ML^{-1}T^{-1})^d$

Equating the indices of M,L and T on both sides , we get

M] $c + d = 1$

$L]$ $a + b - 3c - d = -2$

$T]$ $-a - d = -2$

Step-3-:- there are three equations and four unknowns. Solving these equations in terms of the unknown d, we have

$a = 2 - d$

$b = -d - 1$

$c = 1 - d$

Hence, we can be written

$\frac{\Delta p}{l} = K(V^{2-d} D_h{}^{-d-1} \rho^{1-d} \mu^d)$

$$\frac{\Delta p}{l} = K\left(\frac{V^2\rho}{D_h}\right)\left(\frac{\mu}{VD_h\rho}\right)^d$$

$$\frac{\Delta p D_h}{l\rho V^2} = K\left(\frac{\mu}{VD_h\rho}\right)^d$$

Ex.5

Write the equation of displacement for a free fouling body in time T. Assuming that the displacement dependent on weight, acceleration gravity and time.

Sol.

$S = F(W, g, T) \qquad displacment$

$S = K\, W^a g^b T^c$

The equation must be homogenate in dimension

$M^0L^1T^0 = K\left((MLT^{-2})^a(LT^{-2})^b(T)^c\right)$

Equating the indices of same dimension of quantities

$a = 0$

$a + b = 1 \qquad$ b=1

$-2a - 2b + c = 0 \qquad \therefore c = 2$

$\therefore S = K(W^0 g T^2) or \qquad s = KgT^2$

Ex.6

Find the relation of Reynolds number by dimensional Analysis if Re=F(ρ,μ,V,L)

Sol.

$R_e = F(\rho, \mu, V, L)$

$R_e = K\, \rho^a \mu^b V^c L^d$

$M^0L^0T^0 = K\,(ML^{-3})^a\,(ML^{-1}T^{-1})^b(LT^{-1})^c(L)^d$

$a + b = 0 \qquad a = -b$

$-3a - b + c + d = 0 \qquad d = -2b - c = -2b + b = -b$

$-b - c = 0 \qquad b = -c$

$c = -b$

$\therefore\; R_e = K\, \rho^{-b}\, \mu^b\, V^{-b}\, L^{-b}$

$R_e = K\,\left(\frac{\mu}{\rho VL}\right)^b \qquad K = 1, b = -1$

Ex.7

Find the dynamic pressure over a submerged body due to the flow of uncompressible fluid. Assuming the pressure is function of density and velocity.

Sol.

$p = F(\rho, V)$

$p = K\ \rho^a\ V^b$

$F^1 L^{-2} T^0 = (F^a\ T^{2a}\ L^{-4a})(L^b\ T^{-b})$

From above

1=a , -2=-4a+b, 0=2a-b

$\therefore a = 1\,, b = 2$

$p = K\rho V^2$

Ex.8

Find the expression for the input power to a fan. By dimension analysis, assuming the input power depends on the air density, velocity, viscosity, fan diameter, rotation speed and sound velocity.

Sol.

$\mathrm{power} = \mathrm{K}\left(\rho^a d^b V^c\ \omega^d\ \mu^e\ i^f\right)$

By using (mass, length, time) as fundamental units.

$\mathrm{ML^2T^{-3}} = (\mathrm{ML^{-3}})^a(\mathrm{L})^b(\mathrm{L\,T^{-1}})^c\ (\mathrm{T^{-1}})^d\ (\mathrm{ML^{-1}T^{-1}})^e(\mathrm{L\,T^{-1}})^f$

1=a+e a=1-e

$2 = -3a + b + c - e + f$ then b = 5-2e-c-f

$-3 = -c - d - e - f$ d= 3-c-e-f

Subsititute in power Eqn.

$\mathrm{power} = \mathrm{K}\ \rho^{1-e}\ d^{5-2e-c-f}\ V^c\ \omega^{3-c-e-f}\ \mu^e\ i^f$

$power = K\left[\left(\frac{\rho d^2\omega}{\mu}\right)^{-e}\ \left(\frac{d\omega}{V}\right)^{-c}\ \left(\frac{d\omega}{i}\right)^{f}\right]\omega^3 d^5\rho$

The terms between brackets dimension less

$1^{st}\ term = R_e$

V= Rω

$2^{nd}\ term = \ fan\ ratio.$

$3^{nd}\ term = Mach\,number.$

6.7 Buckingham's Pi Theorem. (Method-2)

Assume, a physical phenomenon is described by

n = number of independent variables like x_1, x_2, $x_3....x_n$ the phenomenon may be expressed as

$$F(x_1, x_2, x_3, \dots \dots x_m) = 0 \tag{6.19}$$

m = number of fundamental dimensions like mass, time, length and temperature or force, length, time and temperature.

Buckingham's theorem defining as the phenomenon can be described in terms of (***n-m***) independent dimensionless group like $\pi_1, \pi_2, \dots \dots \dots \pi_{m-n}$ where $\boldsymbol{\pi}$ terms, represent the dimensionless parameters and consist of different combinations of a number of dimensional variables out of the ***n*** independent variables.

Therefore Eq.(6.19) can be reduced to

$$F(\pi_1, \pi_2, \dots \dots ., \pi_{m-n}) = 0 \tag{6.20}$$

6.7.1 Mathematical Description of (π) Pi Theorem.

A physical problem described by ***n*** number of variable involving ***m*** number of fundamental dimensions (***m*** < ***n***) leads to a system of ***m*** linear algebraic equations with ***n*** variables of the from

$$\left.\begin{aligned} \mathrm{a_{11}x_1} + \mathrm{a_{12}x_2} + \cdots \mathrm{a_{1n}x_n} &= \mathrm{b_1} \\ \mathrm{a_{21}x_1} + \mathrm{a_{22}x_2} + \cdots \mathrm{a_{2n}x_n} &= \mathrm{b_2} \\ \mathrm{a_{m1}x_1} + \mathrm{a_{m2}x_2} + \cdots \mathrm{a_{mn}x_n} &= \mathrm{b_m} \end{aligned}\right\} \tag{6.21}$$

Therefore all feasible phenomena are define with n > m

No physical phenomena is represent

n< ***m*** no solution

m=***n*** one solution

All the parameter have fixed value.

In a matrix from

$$AX=b \tag{6.22}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad and \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

6.7.2 Procedure for Determination π Terms.

m = Number of fundamental dimensions like mass, (M), time (T), Length (L), temperature(θ)

n = number of independent variables or quantities included in physical problem such as (A_1,A_2,A_3-----A_n) where A_1,A_2,A_3-----A_n as pressure , viscosity and velocity, can also be expressed as

$$F1(A_1, A_2, A_3, ----A_n) = 0 \quad (6.23)$$

(***n-m***) = number of dimensionless parameter (π) like $\pi_1, \pi_2, \pi_3, \dots \pi_{n-m}$

π , is represent the dimensionless parameters and consist of different combinitions of a number of dimensional variable. Mathematically, if any variable A_1, depends on independent variable $A_2.A_3\dots A_n$ the function $A_1 = F(A_2, A_3, \dots A_n)$

According to π-theorem, Eq. (6.23) can be written in terms of π- terms (dimensionless groups). Therefore the above equation can reduced to

$$F_1(\pi_1, \pi_2, \dots \pi_{n-m}) = 0 \quad (6.24)$$

The method of determining π parameters is

- Select (***m***) of the (A) quantities with different dimensions
- The above selection which contains among them (***m***) dimensions
- Using the (***m***) selection as repeating variables together with one of the other A quantities for each (π). Each π-term contains (***m+1***) variables.

Note-1, It is essential that no one of the ***m*** selected quantities used as repeating variable be derived from the other repeating variables.

Note-2, Let A_1,A_2,A_3 contain M,L and T, not necessarily in each one , but collectively.

Then the first π parameter is made up as

$$\left.\begin{array}{l} \pi_1 = A_2^{x1} A_3^{y1} A_4^{z1} A_1 \\ \text{The second } \pi_2 = A_2^{x2} A_3^{y2} A_4^{z2} A_5 \\ \text{And so on until } \pi_{n-m} = A_2{}^{x_{n-m}} A_3{}^{y_{n-m}} A_4{}^{z_{n-m}} A_n \end{array}\right\} \quad (6.25)$$

In a bove eqn's the exponents are to be determined

- The dimensions of (A) quantities are substituted
- The exponents of M,L and T are set equal to zero in π parameters

There produce three equations in three unknowns for each π parameter, so that the x,y,z exponents can be determined , and Hence the π parameter.

6.7.3 Selection of Repeating Variables (R.V).

1- (***m***) repeating variable must contain jointly all the fundamental dimension
2- The repeating variable must not form the non-dimensional parameter among them.
3- As far as possible, the dependent variable should not be selected as repeating variable
4- No two repeating variables should have the same dimensions
5- The repeating variables should be chosen in such a way that one variable contains geometric property (e.g , length , L, diameter, d, height, h) , other variable contains flow property (e.g velocity V, acceleration a) and the third variable contains fluid property (e.g mass density ρ, weight density W , dynamic viscosity μ)

(i) L,V,ρ
(ii) d,V,ρ
(iii) l, V,m
(iv) d,V,μ

Ex.9

Show that the lift force F_l on airfoil can be express as $F_L = \rho V^2\ d^2 \emptyset\ (\frac{\rho V d}{\mu}, \propto)$

Where ρ= mass density , V = velocity of flow
μ= dynamic viscosity ∝= Angle of incidence
d = A characteristic depth

Sol.

Left force F_L is function of;ρ, V, d, m, ∝ mathematically, $F_L = f(\ \mu, V, d, \rho, \propto) - - - -(i)$

Or $F_1(\ F_L, \rho\ V, d\ \mu, \propto) = 0 - - - - - - - - (ii)$

∴ Total number of variable, we have ***n***=6

Writing dimensions of each variable

$F_L = M\ L\ T^{-2}, \rho =\ M\ L^{-3}, V = LT^{-1}, d = L, \mu = ML^{-1}T^{-1}\ , \propto = M^0 L^0 T^0$

Thus, number of fundamental dimensions, ***m***=3

$\therefore Number\ of\ \pi - terms = n - m = 6 - 3 = \mathbf{3}$

Eq. (ii) can be written as : $F_1(\pi_1, \pi_2, \pi_3) = 0 - - - - - (iii)$

Each π-term contains (***m+1***) variables, where ***m***=3 and also equal to repeating variables (R.V). Choosing (d, V, ρ) as R.V

$\pi_1 = d^{a1}.V^{b1}.\rho^{c1}.F_L$

$\pi_2 = d^{a2}.V^{b2}.\rho^{c2}.\mu$

$\pi_3 = d^{a3}.V^{b3}.\rho^{c3}.\propto$

$\pi - term$:

$\pi_1 = d^{a1}.V^{b1}.\rho^{c1}.F_L$

$M^0L^0T^0 = L^{a1}(L\,T^{-1})^{b1}(ML^{-3})^{c1}(MLT^{-2})$

Equating the exponents of M,L,T respectively, we get

$for\ M: 0 = c_1 + 1 -----\rightarrow\ c_1 = -1$

$for\ L: 0 = a_1 + b_1 - 3c_1 + 1$

$for\ T: 0 = -b_1 - 2 ---\rightarrow\ b_1 = -2$

$\therefore\ a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Substituting the values of $a_1, b_1\ and\ c_1 in\ \pi_1$, we get

$\pi_1 = d^{-2}.V^{-2}.\rho^{-1}.F_L = \frac{F_L}{\rho V^2 d^2}$

$\pi_2 - term$:

$\pi_2 = d^{a_2}.V^{b_2}.\rho^{c_2}.\mu$

$M^0L^0T^0 = L^{a_2}.(LT^{-1})^{b2}.(ML^{-3})^{c_2}.(ML^{-1}T^{-1})$

Equating the exponents of M,L and T respectively , we get

$for\ M: 0 = c_2 + 1 ---\rightarrow\ c_2 = -1$

$FOR\ L: 0 = a_2 + b_2 - 3c_2 - 1$

For $T: 0 = -b_2 - 1 ---\rightarrow b_2 = -1$

$\therefore\ a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Substituting the values of $a_2, b_2, and\ c_2\ in\ \pi_2, we\ get$

$\pi_2 = d^{-1}.V^{-1}.\rho^{-1}.\mu = \frac{\mu}{\rho V d}$

$or\ \pi_2 = \frac{\rho V d}{\mu}$

$\pi_3 - term$

$\pi_3 = d^{a_3}.V^{b_3}.\rho^{c_3}.\propto$

$M^0L^0T^0 = L^{a_3}.(LT^{-1})^{b_3}.(ML^{-3})^{c_3}.(M^0L^0T^0)$

Equating the exponents of M, L and T respectively , we get

$for\ M: 0 = c_3 + 0 ---\rightarrow\ c_3 = 0$

$for\ L: 0 = a_3 + b_3 - 3c_3 + 0$

$for\ T: 0 = -b_3 + 0 ---\rightarrow b_3 = 0$

$\therefore\ a_3 = 0$

$\therefore\ \pi_3 = d^0.v^0\rho^0.\propto=\propto$

Substituting the values of $\pi_1, \pi_2\ and\ \pi_3\ in\ Eq.(iii), we\ get$

$f_1\left(\frac{F_L}{\rho V^2 d^2}, \frac{\rho V d}{\mu}, \propto\right) = 0$

$\frac{F_L}{\rho V^2 d^2} = \emptyset\left(\frac{\rho V d}{\mu}, \propto\right)$

$or\quad F_L = \rho V^2 d^2\ \emptyset\left(\frac{\rho V d}{\mu}, \propto\right)$

Ex.10

The discharge through a horizontal capillary tube is thought to depend upon the pressure drop per unit length, the diameter and the viscosity. Find the form of the discharge equation.

Sol.

Quantity	Dimensions
Discharge Q	$L^3\ T^{-1}$
Pressure drop/length Δp/l	$M\ L^{-2}T^{-2}$
Diameter D	L
Viscosity μ	$M\ L^{-1}T^{-1}$

Then $F\left(Q, \frac{\Delta p}{L}, D\ , \mu\right) = 0$

Three dimension are used, and with four quantities these will be one π parameter

$\boldsymbol{n} = 4\ , \boldsymbol{m} = 3 - - - - \rightarrow\ \pi = n - m = 4 - 3 = 1$

$\pi = \ Q^{X_1}(\frac{\Delta p}{l})^{Y_1} D^{Z_1}\mu$

Substituting in the dimension gives

$\pi = (L^3\ T^{-1})^{X_1}\ (ML^{-2}T^{-2})^{Y_1}\ (L^{Z_1})(M\ L^{-1}T^{-1}) = M^0L^0T^0$

The exponents of each dimension must be the same on both sides of the equation.

With L; $3X_1 - 2Y_1 + Z_1 - 1 = 0 - - - - - - -(i)$

M; $Y_1 + 1 = 0 - - - - \rightarrow\ Y_1 = -1$

T; $-X_1 - 2Y_1 - 1 = 0 - - - - - - - \rightarrow X_1 = 1$

From Eq. (i) $Z_1 = -4$

$\pi = Q \frac{\mu}{\left(\frac{\Delta p}{l}\right) D^4}$

After solving for Q

$Q = C \frac{\Delta p}{L} \frac{D^4}{\mu}$

Ex.11

Consider pressure drop in a tube of length l, hydraulic diameter d, surface roughness ∈, with fluid of density ρ and viscosity μ moving with average velocity U. Using Buckingham's π - theorem obtain an expression for Δp .

Sol.

This can be expressed as

$f(\Delta p, U, d, l, \in, \rho, \mu) = 0$

Now ***n***=7 since the phenomenon involves 7 independent parameters.

We select ρ, U, d as repeating variables (so that all 3 dimensions are represented)

Now, 4 π ----→(7-3) parameters are determined as

$\pi_1 = \rho^{a_1} U^{b_1} d^{c_1} \Delta p$

$\pi_2 = \rho^{a_2} U^{b_2} d^{c_2} \mu$

$\pi_3 = \rho^{a_3} U^{b_3} d^{c_3} l$

$\pi_4 = \rho^{a_4} U^{b_4} d^{c_4} \in$

Now basic units

$\rho - - - - \rightarrow ML^{-3}$

$U - - \rightarrow LT^{-1}$

$d - - - \rightarrow L$

$\Delta p - - - \rightarrow ML^{-1}T^{-2}$

$\mu - - - - \rightarrow ML^{-1}T^{-1}$

$\in - - - \rightarrow L$

$l - - - \rightarrow . L$

All π parameters $- - - - \rightarrow M^0 L^0 T^0$

$a_1 = -1;\ b_1 = -2\ ;\ C_1 = 0$

$a_2 = -1;\ b_2 = -1\ ;\ c_2 = -1$

$a_3 = 0;\ b_3 = 0;\ c_3 = -1$

$a_4 = 0;\ b_4 = 0;\ c_4 = -1$

Thus writing $\pi_1 = f(\pi_2, \pi_3, \pi_4)$

∴ The π group can be written as follows,

$\pi_1 = d^0 V^{-2} \rho^{-1} \Delta p = \frac{\Delta p}{\rho V^2}$ $\boldsymbol{Eu\ \ .No.}$

$\pi_2 = \frac{\mu}{dV\rho}\ or \frac{dV\rho}{\mu}$ $\boldsymbol{Re\ \ .N_o.}$

$\pi_3 = d^{-1} V^0 \rho^0 l = \frac{l}{d}$

$\pi_4 = \frac{\in}{d}$

∴ The new relation can be writing

$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{dV\rho}{\mu}, \frac{l}{d}, \frac{\in}{d}\right) = 0$

When conclude Δp

$\frac{\Delta p}{\rho V^2} = f_2\left(Re\ \ , \frac{l}{d}, \frac{\in}{d}\right)$

$\rho = \frac{W}{g}$

$\frac{\Delta p}{W} = \frac{V^2}{2g} . f_2\left(Re\ \ , \frac{l}{d}, \frac{\in}{d}\right)$

The pressure drop is function of (L/d) exponent to(1) in darcy equation

$\frac{\Delta p}{W} = \frac{V^2}{2g} . \frac{L}{d} . f_3\left(Re\ \ , \frac{\in}{d}\right)$

Therefore

$\frac{\Delta p}{W} = (Factor\ \boldsymbol{f})\left(\frac{L}{d}\left(\frac{V^2}{2g}\right)\right)$

<u>Ex.12</u>

Assume the input power to a pump is depend on the fluid weight per unit volume, flow rate and head produced by the pump. Create a relation by dimensional analysis between the power and other variables by two methods.

<u>Sol.</u>

Method-1

$P = f(\, W, Q, H)$

$P = K\, W^a Q^b H^c$

In Dimension analysis

$F^1 L^1 T^{-1} = (FL^{-3})^a (L^3 T^{-1})^b (L)^c$

Hence

$a = 1\ , 1 = -3a + 3b + c$

$\therefore a = 1, b = 1, c = 1$

$P = KWQH$

Method-2

$F(P, W, Q, H) = 0$

The variables in dimensions are

$\mathrm{P} - - - \rightarrow \mathrm{FLT}^{-1}$

$\mathrm{Q} - - \rightarrow \mathrm{L}^{3}\mathrm{T}^{-1}$

$\mathrm{W} - - \rightarrow \mathrm{FL}^{-3}$

$\mathrm{H} - - - \rightarrow \mathrm{L}$

The four variables in 3 fundamental dimensional $\therefore\ \pi\ group\ is (4-3) = 1$

Choice Q, W, H as variable with unknown exponent

$\therefore\ \pi_1 = (Q)^{a_1}\ (W)^{b_1}\ (H)^{c_1}\ P$

$or\ \pi_1 = (L^{3a_1}\ T^{-a_1})(F^{b_1}\ L^{-3b_1})(L^{c_1})(F\ L\ T^{-1}) = F^0 L^0 T^0$

Exponent equality foe F,L,T producing

$\mathrm{a}_1 = -1, \mathrm{b}_1 = -1, \mathrm{c}_1 = -1$

$\therefore\ \pi_1 = \mathrm{Q}^{-1}\ \mathrm{W}^{-1}\mathrm{H}^{-1}\ \mathrm{P} = \frac{\mathrm{P}}{\mathrm{QWH}}$

$\pi_1 = F\left(\frac{P}{QWH}\right)$

$P = K(QWH)$

<u>Ex.13</u>

Assume the input power to a pump is depend on the fluid weight (W), flow rate (Q) and head produced by pump (H), create a relation by dimension analysis between the power input and other variables by using FLT system.

<u>Sol.</u>

<u>Step-1</u>

Quantities	Dimensions
Power (P)	$\mathrm{F\ L\ T}^{-1}$
Flow rate (Q)	$\mathrm{L}^3\mathrm{T}^{-1}$
Weight(W) per unit volume	$F\ L^{-3}$
Head (H)	L

<u>Step-2:-</u> there are four variables & (3) fundamental dimensions

$\therefore\ \pi\ group\ is\ (4-3) = 1$

Step-3:

Choice Q, W, H as variable with unknown exponent.

$F_1(P, W, Q, H) = 0 \longrightarrow F_2(\pi_1) = 0$

$\therefore\ \pi_1 = (Q)^{a_1}\ (W)^{b_1}(H)^{c_1}\ P$

Or $\pi_1 = (L^3T^{-1})^{a_1}\ (F\ L^{-3})^{b_1}(L)^{C_1}\ (FLT^{-1})^1 = F^0L^0T^0$

F] $b_1 + 1 = 0 - -\rightarrow\ b_1 = -1$

L] $3a_1 - 3b_1 + c_1 + 1 = 0 \quad \rightarrow 3a_1 + c_1 = -4$

T] $-a_1 - 1 = 0 - -\rightarrow\ a_1 = -1$

$\therefore\ c_1 = -1$

$\therefore\ \pi_1 = Q^{-1}W^{-1}H^{-1}\ P = \frac{P}{QWH}$

$\pi_1 = f\left(\frac{P}{QWH}\right)$

$P = K(Q, W, H)$

Ex.14

Assuming the resistant force for a body submerged in a fluid is function of (density ρ , velocity V, viscosity μ and characteristic length L). Conclude a general equation of resistant force by using FLT system.

Sol.

Step.1

Quantities	Dimension
Force(F)	F
Density(ρ)	$F\ L^{-4}T^2$
Velocity(V)	$L\ T^{-1}$
Viscosity(μ)	$F\ L^{-2}T$
Length(L)	L

$F_1(F, \rho, V, L, \mu) = 0 \quad n = 5\ , m = 3$

Step-2

We have 5 variables with 3 fundamental dimensions

$\therefore\ \pi\ groups = 5 - 3 = 2$

We choice 3 repeated variables of unknown exponents

$\pi_1 = (L)^{a_1}\ (V)^{b_1}\ (\rho)^{c_1}\ F = (L)^{a_1}\ (LT^{-1})^{b_1}\ (F\ T^2L^{-4})^{c_1}\ (F) = F^0L^0T^0$

F] $\quad C_1 + 1 = 0 - -\rightarrow\ C_1 = -1$

L] $a_1 + b_1 - 4c_1 = 0 --\rightarrow\ a_1 + b_1 = -4$

T] $-b_1 + 2C_1 = 0-\rightarrow\ b_1 = -2;\ \therefore\ a_1 = -2$

$\therefore\ \pi_1 = L^{-2}\, V^{-2} \rho^{-1} F = F/L^2V^2\rho$

$\pi_2 = (L)^{a_2}\,(V)^{b_2}\,(\rho)^{C_2}\,\mu = (L)^{a_2}\,(L\,T^{-1})^{b_2}\,(F\,L^{-4}T^2)^{C_2}\,(F\,L^{-2}T) = F^0L^0T^0$

F] $C_2 + 1 = 0 --\rightarrow\ C_2 = -1$

L] $a_2 + b_2 - 4C_2 - 2 = 0$

$a_2 + b_2 + 2 = ---\rightarrow\ a_2 + b_2 = -2$

T] $-b_2 + 2C_2 + 1 = 0 --\rightarrow\ b_2 = -1$

$\therefore\ a_2 = -1$

$\therefore\ \pi_2 = L^{-1}V^{-1}\rho^{-1}\mu = \frac{\mu}{LV\rho} --\rightarrow \pi_2^{-1} = R_e$

$\therefore f(\pi_1, \pi_2) = 0$

$F_1(\pi_1, \pi_2^{-1}) = 0 --\rightarrow\ F_1\left(\frac{F}{L^2V^2\rho}, R_e\right) = 0$

$\therefore F = L^2V^2\rho\, F_2(R_e)$

Problems.

P6.1 A stationary sphere in water moving at velocity of ***1.6 m/s*** experiences a drag of ***4 N***. Another sphere of twice the diameter is placed in a wind tunnel. Find the velocity of the air and the drag which will give dynamically similar conditions. The ratio of kinematic viscosities of air and water is ***13***, and the density of air is $1.28 kg/m^3$. ***[V_{air}=10.4m/s, F_d=0.865 N]***

P6.2 A 1:80 scale model of an aircraft was tested in air at ***20C°*** moves with speed ***40m/s***

a- What is the speed model if its submerged in water at 26C°,

$$\nu_{air} = 14.86 * 10^{-6} \frac{m^2}{s};\ \nu_{water} = 0.864 * 10^{-6} \frac{m^2}{s}$$ ***[V=2.325 m/s]***

b- Determine the air resistance to the prototype if model water resistance is ***7.43N***. ***[F_p=2.643 N]***

P6.3 Predicting the general form of input power to a fan , which is depends on the density, velocity, viscosity of air, diameter, angular velocity of fan and speed of sound ***i***. $\left[\boldsymbol{P = k\left(\left(\frac{\omega d}{V}\right)(Re)(Ma)\right)\rho V^3 d^2}\right]$

P6.4 Write the equation of displacement ***(S)*** for a free failing body with time ***(T).*** Assuming that the displacement depends on weight ***(W)***, gravity ***(g)*** and time***(T)***. ***[$S=kgT^2$]***

P6.5 A model spillway has a flow of ***100 l/s*** per meter of width. What is the actual flow for the prototype spillway if the model scale is ***1:20***

[q_p=8.94 m^3/s per m]

P6.6 Researchers plan to test a ***1:13*** model of a ballistic missile in a high – speed wind tunnel. The prototype missile will travel at ***380m/s*** through air at ***23C°*** and ***95.0 kPa*** absolute

(a) If the air in the tunnel test section has a temperature -20C° at a pressure of 89 kPa absolute, what its velocity? ***[V_m=351 m/s]***

b) Estimate the drag force on the prototype if the drag force on the model is ***400N.*** ***[F_p=72310 N]***

P6.7 Derive an expression for the shear stress at the pipe wall when an incompressible fluid flows through a pipe under pressure. Use diameter ***D***, flow velocity ***V***, viscosity $\boldsymbol{\mu}$ and density $\boldsymbol{\rho}$ of the fluid by using $\boldsymbol{\pi}$ theorem. $[\boldsymbol{\tau = \rho V^2 \varphi(Re)}]$

P6.8 Derive an expression for the drag on aircraft flying at supersonic speed, in the form of a function including dimensionless quantities by using Buckingham's (π) theorem. $[\boldsymbol{F_d = \rho L^2 V^2 f_1(Re, Ma)}]$

P6.9 Derive an expression for small flow rates over a spillway in the form of a function including dimensionless quantities. Use dimensional Analysis with the following parameters height of spillway $\boldsymbol{y}$, head on the spillway $\boldsymbol{H}$, viscosity of liquid $\boldsymbol{\mu}$, density of liquid $\boldsymbol{\rho}$, surface tension $\boldsymbol{\sigma}$ and acceleration due to gravity $\boldsymbol{g}$. $\left[\boldsymbol{q = g^{1/2} H^{3/2} f_2 \left(\frac{h}{y}, Re, We\right)}\right]$

P6.10 The resisting force $\boldsymbol{F}$ at a plane during flight can be considered as dependent up on the length of aircraft $\boldsymbol{L}$ velocity $\boldsymbol{V}$ air viscosity$\boldsymbol{\mu}$, air density ρ and black modules of air $\boldsymbol{k}$. express the functional relationship between these variables and resisting force using dimensional analysis. $\boldsymbol{F = L^2 V^2 \rho * \varphi \left(\frac{\mu}{LV\rho}, \frac{k}{V^2\rho}\right)}$

P6.11 The pressure difference $\boldsymbol{\Delta p}$ in a pipe of diameter $\boldsymbol{D}$ and length $\boldsymbol{L}$ due to turbulent flow depends on the velocity $\boldsymbol{V}$, viscosity $\boldsymbol{\mu}$, density $\boldsymbol{\rho}$ and roughness $\boldsymbol{\varepsilon}$ using Buckingham's $\boldsymbol{\pi}$ theorem to obtain an expression for $\boldsymbol{\Delta p}$. $[\Delta \boldsymbol{p} = \boldsymbol{\frac{\mu V}{D} * \frac{L}{D} * \left(\frac{\rho V D}{\mu}\right)}]$

P6.12 Prove that the shear stress $\boldsymbol{\tau}$ in a fluid flowing through a pipe can be expressed by the equation $\tau = \rho V^2 \emptyset(\frac{\mu}{\rho D V})$

Where; $\boldsymbol{D}$ = diameter, $\boldsymbol{\rho}$ = mass density, $\boldsymbol{V}$= velocity $\boldsymbol{\mu}$ = viscosity.

P6.13 A model of a submarine of scale ***1/40*** is tested in wind tunnel. Find the speed of air in wind tunnel if the speed of submarine in sea water is ***15m/s***. Also find the ratio of the resistance between the model and its prototype. Take the value of kinematic viscosities for sea water and air as ***0.012*** stokes and ***0.016*** stokes respectively. The weight density of sea water and air are given as ***10.1kN/m³*** and ***0.0122 kN/m³*** respectively. $\boldsymbol{V_m = 800\frac{m}{s}, \frac{R_m}{R_p} = 0.00214}$

P6.14 A spillway model is to be built to a geometrically similar scale of ***1/50*** across a flow of ***600m*** width. The prototype is ***15m*** high and maximum head on it is expected to be ***1.5m***.

i) What height of model and what head on model should be used.

[H_m=0.3 m]

ii) If flow over the model for a particular head is ***12 L/s*** what flow per meter length at prototype is expected. $[\boldsymbol{\frac{Q_p}{L_p} = 7071\frac{lit}{s}}]$

P6.15 In an airplane model of size ***1/50*** of its prototype the pressure drop is ***4 bar***. The model is tested in water. Find the corresponding pressure drop in the prototype. Take specific weight of air = 0.00124kN/m^3. The viscosity of water as ***0.01 poise*** while the viscosity of air is ***0.00018 poise***. ***[Δp=0.0004114 bar]***

P6.16 An oil of ***S.G. 0.9*** and viscosity ***0.003 N.s/m²*** is to be transported at the rate of ***3000 L/s*** through a ***1.5m*** diameter pipe. Test was conducted on a ***15cm*** diameter pipe. Using water at ***20C°*** if the viscosity at $20C^{\circ}$ is ***0.001 N.s/m²*** fined.

i) Velocity of flow in the model. ***[V_m=5.1 m/s]***
ii) Rate at flow in the model. ***[Q=80.9 lit/s]***

P6.17 A geometrically similar model of an air duct is built to ***1/25*** scale and tested with water which is ***50*** times more viscous and ***800*** times than air. When tested under dynamically similarity conditions the pressure drop is 2 bar in the model. Find the corresponding pressure drop in full scale prototype. ***[1.024*10^{-3}bar]***

P6.18 In a geometrically similar model of spillway the discharge per length is ***0.2m³/s***. If the scale of the model is ***1/36***, find the discharge per meter run of the prototype. ***[q_p=43 m³/s]***

P6.19 The force required to tow a 1:30 scale model of a modern boat in a lake at a speed of ***2m/s*** is ***0.5N***. Assuming that the viscous resistance due to water and air is negligible in comparison with the wave resistance calculate the corresponding speed of the prototype for dynamically similar conditions. What would be the force required to propel the prototype at that velocity in the same lack? ***[F_p=13500 N]***

P6.20 In an airplane model at size ***1/40*** of its prototype the pressure drop is ***7.5 kN/m²*** the model is tested in water. Find the corresponding pressure drop in the prototype. Take density of air = ***1.24 kg/m³***, density of water = ***100 kg/m³***, Viscosity of air = ***0.00018 poise***, Viscosity of water = ***0.01 poise***. ***[Δp_p=1.225 N/m²]***

CHAPTER 7

Viscous Incompressible Flows in Pipes

Part-One
(Laminar Flow)

7.1 *Introduction.*

Real fluids possess viscosity, while ideal fluid is inviscid. The viscosity of fluid introduce resistance to motion by developing shear or frictional stress between the fluid layers and between fluid layers and the boundary, which causes the real fluid to a adhere to the solid boundary and hence no relative motion between fluid layer and solid boundary.

Viscosity causes the flow to occur in two modes laminar and turbulent flow. Reynolds number < 2000, the flow is always laminar through a pipe which is critical value of ***Re*** for circular pipe. Flow between parallel plates based on mean velocity and distance between the plates.

$$Re \quad = \frac{Inertia force}{viscous force}$$

The flow is laminar when one of the conditions occurs

i) Viscosity is very high.
ii) Velocity is very low.
iii) The passage is very narrow.

7.2 *Relationship between Shear Stress and Pressure Gradient.*

The shear stress is maximum at the boundary and gradually decreases with increase in distance from the solid boundary where the velocity is zero at the boundary. A pressure gradient exists which overcome the shear resistance and causes the fluid to flow. Due to non uniform distribution of velocity, the fluid at any layer moves at a higher velocity than the layer below.

The motion of the fluid element will be resisted by shearing or frictional force which must be overcome by maintaining a pressure gradient in the direction of flow, from fig 7.1, Let τ = shear stress on the lower face ABCD of the element

$\tau + \frac{\partial \tau}{\partial y}\delta y =$ Shear stress on the upper face $\acute{A}\acute{B}\acute{C}\acute{D}$ of the element.

For 2-dimensional steady flow there will be no shear stress on the vertical faces ABB'A' & CDD'C' as in Fig. 7.1. Thus the only forces acting on the element in the direction of flow (x-axis) will be the pressure and shear forces. Let $\delta x, \delta y\ and\ \delta z$ are element thickness in x, y and z directions.

Net shearing force on the element in y-direction is equal to

$$= \left(\tau + \frac{\partial \tau}{\partial y}\ \delta y\right)\delta x.\delta z - \tau.\delta x\delta z = \frac{\partial \tau}{\partial y}\delta x.\delta y.\delta z \qquad (7.1)$$

Net pressure force on the element in x-direction is equal to

$$= \ p.\delta y.\delta z - \left(p + \frac{\partial p}{\partial x}\delta x\right)\delta y.\delta z = -\frac{\partial p}{\partial x}\delta x.\delta y.\delta z \qquad (7.2)$$

For the flow to be steady and uniform these begin no acceleration, the sum of the forces must be zero, from (7.1 & 7.2)

$$\frac{\partial \tau}{\partial y}.\delta x.\delta y.\delta z - \frac{\partial p}{\partial x}.\delta x.\delta y.\delta z = 0$$

The relationship between shear stress and pressure gradient is

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y} \qquad (7.3)$$

Eq. (7.3) indicates that the pressure gradient in the direction of flow is equal to the shear gradient in the direction normal to the direction of flow. This is applicable for laminar and turbulent flow.

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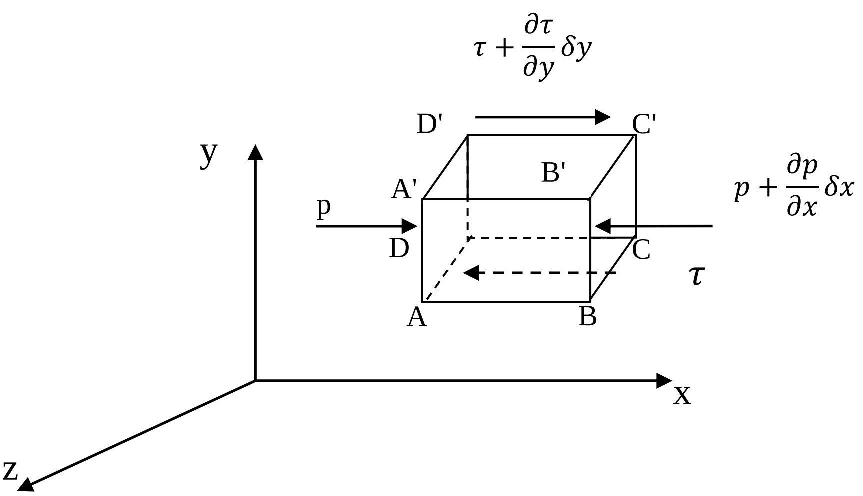


Figure 7.1: Pressure and Shear stress Forces on a Fluid Element.

7.3 Laminar Flow between Parallel Plates.

Consider two parallel plates with (h) distance apart. For steady flow between them a pressure gradient $\partial p/\partial x$ exist which related shear stress in y-direction.

Since $\tau = \mu \frac{du}{dy}$

Then $\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$

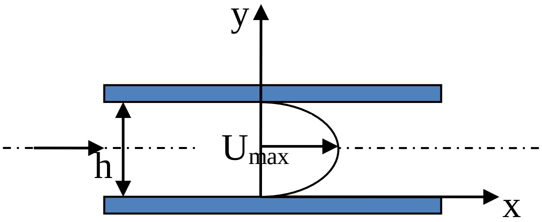


Figure 7.2: Laminar flow between parallel plates.

Integration gives

$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + C_1 y + C_2$

$u = 0 \;\; at\; y = 0 \; and \; y = h$

$C_2 = 0\,, \quad C_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h$

And

$u = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy - y^2)$

This equation is a parabola shape with vertex at center line (y=h/2) when the maximum velocity occurs

$U_{max} = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{h^2}{2} - \frac{h^2}{4}\right)$

Or $u_{max} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) h^2$ (7.4)

The (-ve) of pressure gradient is the pressure drop in the direction of flow. The discharge ***dq*** through a small area of depth ***dy*** per unit width is

dq = udy

$dq = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy - y^2)dy$

$Q = \int_0^h \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy \quad - y^2)dy$

$Q = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\left(h\frac{y^2}{2} - \frac{y^3}{3}\right)\right)_0^h$

$Q = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right) h^3$ (7.5)

Mean velocity of flow $\quad \bar{u} = \frac{Q}{flow\ area} = \frac{Q}{h*1}$

$\bar{u} = \frac{\frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right)h^3}{h} = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right) h^2$ (7.6)

The above velocity ($\bar{u}$) may be used to calculate the pressure drop

$-\partial p = \frac{12\mu\bar{u}}{h^2}\partial x \quad \rightarrow \quad \frac{\partial p}{\partial x} = \frac{12\mu\bar{u}}{h^2}$ (7.7)

From Eq's (7.4 & 7.6)

$U_{max} = \frac{3}{2}\bar{u}$

The pressure drop between two sections with distance $x_1\ and\ x_2$ from origin is

$\int_{p_1}^{p_2}(-dp) = \int_{x_1}^{x_2} \frac{12\mu\bar{u}}{h^2} dx$

$p_1 - p_2 = \frac{12\mu\bar{u}}{h^2}(x_2 - x_1)$

If L is the length between the sections

$$p_1 - p_2 = \frac{12\mu\bar{u}L}{h^2} \tag{7.8}$$

The variation of shear stress in the y-direction is

$$\tau = \mu\frac{\partial}{dy}\left[\frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hy - y^2)\right]$$

$$\tau = \frac{1}{2}\left(-\frac{\partial p}{\partial x}\right)(h - 2y)$$

$$\tau = -\frac{\partial p}{\partial x}\left(\frac{h}{2} - y\right) \tag{7.9}$$

Shearing stress varies linearly with y, it is maximum at the boundary, y=0 and y=h

$$at\ y = 0 \quad \tau_0 = \frac{h}{2}\left[-\frac{\partial p}{\partial x}\right] = \frac{h}{2}\left(\frac{12\mu}{h^2}\bar{u}\right) = \frac{6\mu\bar{u}}{h} \tag{7.10}$$

$$At\ y = h \quad \tau_0 = -\frac{h}{2}\left(-\frac{\partial p}{\partial x}\right) = -\frac{h}{2}\left(\frac{12\mu}{h^2}\bar{u}\right) = -\frac{6\mu\bar{u}}{h} \tag{7.11}$$

Shearing stress at the center y = h/2 is zero

$$\tau = -\frac{\partial p}{\partial x}\left(\frac{h}{2} - \frac{h}{2}\right) = 0$$

Ex.1

Incompressible fluid flows through a rectangular passage of width (b), small depth (t) and length L, in the direction of flow. If the pressure drop between the two ends is p calculate the shear stress at the wall of the passage in terms of mean velocity and the coefficient of viscosity

Sol.

$$p = \frac{12\mu\bar{u}L}{t^2}$$

$$\tau_0 = \frac{t}{2}\left(-\frac{\partial p}{\partial x}\right) = \frac{tp}{2L}$$

Then $\tau_0 = \frac{6\mu\bar{u}}{t}$

Ex.2

Water at 20C° flows between two large parallel plates separated by a distance of 16mm. calculate

i) Max. velocity

ii) Shear stress at the wall if the average velocity is (0.4 m/s) (take μ for water = 0.01 Poise)

Sol.

i) $U_{max} = \frac{t^2}{8\mu}\left(-\frac{\partial p}{\partial x}\right) = \frac{3}{2}\bar{u} = \frac{3}{2} * 0.4 = 0.6\ \frac{m}{s}$

ii) Shear stress at the wall

$$\tau_0 = -\frac{6\mu\bar{u}}{t} = -0.15\ n/m^2$$

7.4 Couette Flow.

Couette flow is the flow between two parallel plates as in Fig. 7.3 one plate is at rest and the other is moving with a velocity U, assuming infinitely large in z-direction

The governing Equation is

$$\frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$$

Flow is independent of any variation in z-direction the boundary condition are

i) At y = 0 , u = 0

ii) At y = h , u = U

After integration twice we get

$$u = \frac{1}{2\mu}\frac{dp}{dx}\, y^2 + c_1 y + c_2$$

At y = 0, u = 0, then $c_2 = 0$

$$u = \frac{1}{2\mu}\frac{dp}{dx}\, y^2 + c_1 y$$

At y=h , u= U

$$\therefore\ C_1 = \frac{U}{\mathrm{h}} - \frac{1}{2\mu}\frac{\mathrm{dp}}{\mathrm{dx}}\,\mathrm{h}$$

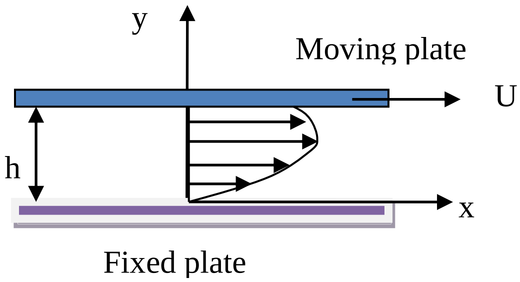


Figure 7.3: Couette flow between parallel plates

Then the expression for ***u*** becomes

$$u = \frac{y}{h}\,U - \frac{1}{2\mu}\frac{dp}{dx}\,(hy - y^2) \qquad (7.12)$$

Multiply and divided by(h^2)

$$\text{Or } u = \frac{y}{h}\,U - \frac{h^2}{2\mu}\,.\frac{dp}{dx}.\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (7.13)$$

Eq. 7.13 can also be expressed in the form

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U}.\frac{dp}{dx}\,.\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

$$\text{Or } \frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (7.14)$$

Where $P = -\frac{h^2}{2\mu U}\left(\frac{dp}{dx}\right)$

P is known as the non- dimensional pressure gradient. When P=0, the velocity distribution across the channel reduced to

$$\frac{u}{U} = \frac{y}{h}\ \textit{is known as simple couette flow}\ .$$

- When P>0 , i.e for a negative pressure gradient (-dp/dx) in the direction of motion , the velocity is positive over the whole gap.
- When P < 0, these is positive or adverse pressure gradient in the direction of motion and the velocity over a portion of channel width can become negative and back flow may occur near the wall, which is at rest

7.4.1 Maximum and Minimum Velocities.

The variation of maximum and minimum velocity in the channel is found out by setting du/dy = 0 from Eq. 7.14, we can write

$$\frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left(1 - 2\frac{y}{h}\right)$$

Setting du/dy = 0 gives

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P} \tag{7.15}$$

By studying Eq. 7.15 we conclude that

1- The maximum velocity for P=1 occurs at y/h =1 and equal to U. For P>1 , the maximum velocity occurs at a location y/h <1.

2- i.e that with P>1, the fluid particles attain a velocity higher than that of the moving plate.

3- for P= -1 , the minimum velocity occurs at y/h =0 for P< -1 , the minimum velocity occurs at a location y/h>1

4- This means that these occurs a back flow near the fixed plate. The values of maximum and minimum velocities can be determined by substituting the value of y from Eq. 7.15 into Eq. 7.14 as

$$\left.\begin{aligned} U_{max} &= \frac{U(1+P)^2}{4\mathrm{P}} \quad \text{for P} \geq 1 \\ U_{min} &= \frac{U(1+P)^2}{4\mathrm{P}} \quad for\ P \leq 1 \end{aligned}\right\} \tag{7.16}$$

The expression for shear stress can be obtained by substituting the value of ***u*** in Newton's equation of viscosity

$$\tau = \mu\frac{du}{dy} = \mu\frac{d}{dy}\left[\frac{yU}{h} - \frac{1}{2\mu}\frac{dp}{dx}(hy - y^2)\right]$$

$$\tau = \mu\frac{U}{h} + \left(-\frac{dp}{dx}\right)\left(\frac{h}{2} - y\right) \tag{7.17}$$

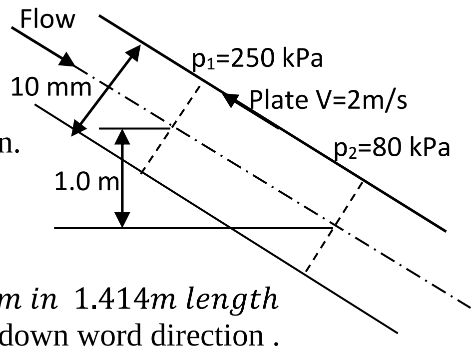
The shear stress at the center is

$$\tau = \mu\frac{U}{h} \tag{7.18}$$

Ex.3

Laminar flow takes place between parallel plates 10 mm apart. The plates are inclined at 45° with the horizontal. For oil of viscosity 0.9 kg/m.s and mass density is 1260 kg/m^3, the pressure at two points 1.0 m vertically apart are 80 kN/m^2 and 250 kN/m^2 when the upper plate moves at 2.00 m/s velocity relative to the lower plate but in opposite direction to flow determine

i) velocity distribution
ii) max. velocity
iii) shear stress on the top plate



Sol.

Consider section 1&2 from Bernoulli's Eqn.

$$\mathrm{H}_1 - \mathrm{H}_2 = -\left(\frac{p_1}{\gamma} + Z_1\right) + \left(\frac{p_2}{\gamma} + Z_2\right)$$

$$= -\left(\frac{250000}{9.806*1260} + 1\right) + \left(\frac{80000}{9.806*1260} + 0\right)$$

$$H_1 - H_2 = -21.234 + 6.475 = -14.759m\ in\ 1.414m\ length$$

Since $H_1$is greater than H_2, flow will be in down word direction .

$$\frac{\partial H}{\partial x} = -\frac{14.759}{1.414} = -10.438$$

And $\frac{\partial p}{\partial x} = \gamma \frac{\partial H}{\partial x} = -10.438 * 1260 * 9.806 = -128.97 \frac{kN/_{m^2}}{m}$

$\frac{u}{U} = \frac{y}{h} - \frac{1}{2\mu U}\frac{\partial p}{\partial x}(yh - y^2)$

$U = -2\frac{m}{s},\ h = 0.01m, \mu = \frac{0.9kg}{ms}$

$\therefore u = -\frac{2}{0.01}y - \frac{1}{2*0.9}\ (-128967.33)(0.01y - y^2)$

i) $u = 516.4364\, y - 71648.5y^2$

To find y at which u is max. set du/dy=0 $= 516.486 - 143297.2y$ or $y = 3.604 * 10^{-3}m$

ii) $\therefore\ u_{max} = (516.486 * 0.003604) -\ (71648.2 * 0.003604^2) = 0.9308\frac{m}{s}$

iii) $\tau_0 =\ \mu\, (\frac{\partial u}{\partial y})_{y=0.01} = 0.9(516.486 - 143297.2 * 0.01) = -824.837\ N/m^2$

7.5 *Pipe of Circular Cross-Section.*

7.5.1 Hagen-Poiseuille Flow.

Consider fully developed laminar flow through a straight tube of circular cross – section as in Fig. 7.4 . Rotational symmetry is considered to make the flow two – dimensional axisymmetry. Let us take ***x-axis*** as the axial of the tube along which all the fluid particles travel, i.e.

$V_x \neq 0, V_r = 0, V_\theta = 0$

Now from continuity equation, we obtain

$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_x}{\partial x} = 0\ \left[for\ rotational\ symmetry, \frac{1}{r}.\frac{\partial v_\theta}{\theta} = 0\right]$

This means $V_x = V_x(r, t)$

Invoking $\left[V_r = 0, V_\theta = 0\ \frac{\partial V_x}{\partial x} = 0, and\, \frac{\partial}{\partial \theta}(\ any\ quantitng) = 0\right]$

With Navier–Stokes equation, we obtain in the x-direction

$$\frac{\partial V_x}{\partial t} = -\frac{1}{\rho}.\frac{\partial p}{\partial x} +\ \nu\left(\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r}.\frac{\partial V_x}{\partial r}\right) \tag{7.19}$$

For steady flow, the governing equation becomes

$$\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r}.\frac{\partial V_x}{\partial r} = \frac{1}{\mu}\frac{dp}{dx} \tag{7.20}$$

The boundary conditions are

i) At $r = 0\ , V_x\ \ is\ finit\ \& \frac{\partial V_x}{\partial r} = 0$

ii) At r = R, V_x=0 yield Eq. 7.20 can be written after multiplying by ***r***

$r\frac{d^2V_x}{dr^2} + \frac{dV_x}{dr} = \frac{1}{\mu}.\frac{dp}{dx}r$

$or\ \frac{d}{dr}\left(r\frac{dV_x}{dr}\right) = \frac{1}{\mu}\frac{dp}{dx}\ r$ by integration

$r\frac{dV_x}{dr} = \frac{1}{2\mu}.\frac{dp}{dx}\ r^2 + A$

$\frac{dV_x}{dr} = \frac{1}{2\mu} \cdot \frac{dp}{dx} r + \frac{A}{r}$ by integration

$V_x = \frac{1}{4\mu} \cdot \frac{dp}{dx} r^2 + A \ln r + B$

At $r = 0$ $V_x = finite$ & $\frac{dV_x}{dr} = 0 \rightarrow A = 0$

$at\ r = R, V_x = 0$

$B = -\frac{1}{4\mu} \cdot \frac{dp}{dx} \cdot R^2$

$\therefore V_x = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right)\left[1 - \frac{r^2}{R^2}\right]$ (7.21)

This shows that the axial velocity profile in a fully developed laminar pipe flow is having parabolic variation along ***r***.

At $r = 0$, $as\ such$, $V_x = V_{x\,max}$

$V_{x\,max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right)$ (7.22)

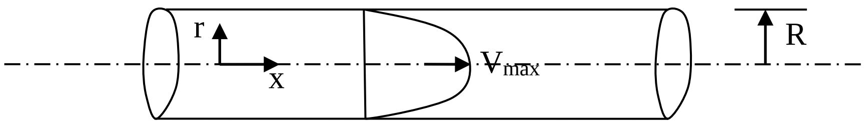


Figure 7.4: Flow in circular pipe.

7.5.2 Volumetric Flow Rate.

The average velocity in pipe is

$V_{av.} = \frac{Q}{\pi R^2} = \frac{\int_0^R 2\pi r\, V_x(r) dr}{\pi R^2}$ substitute Eq. 7.21

$or\ V_{av.} = \frac{\frac{2\pi R^2}{4\mu}\left(-\frac{dp}{dx}\right)\left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right]}{\pi R^2}$

$V_{av.} = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) = \frac{1}{2} V_{x\,max} \quad \rightarrow V_{x\,max} = 2V_{av}$ (7.23)

Now, the discharge ***Q*** through a pipe is given by

$Q = \pi R^2 V_{av}$ (7.24)

$Q = \pi R^2 \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right)$

$or\ Q = -\frac{\pi d^4}{128\mu}\left(\frac{dp}{dx}\right)$ (7.25)

From Eq's 7.22 & 7.23

$\frac{p_1 - p_2}{L} = 4\, V_{max} \frac{\mu}{R^2} = 32\mu \frac{V_{av}}{d^2}$ (7.26)

Eq. 7.26 is known as the ***Hagen- Poiseuille*** equation.

Ex.4

Oil mass density is 800 kg/m^3 and dynamic viscosity is 0.002 kg/m.s flow through 50mm diameter, pipe length is 500 m and the discharge flow rate is 0.19*10^{-3} m^3/s determine

i) Reynolds number of flow.
ii) Center line velocity.
iii) Loss of pressure in 500 m length.
iv) Pressure gradient.
v) Wall shear stress.

Sol.

$$V_{av.} = \frac{4Q}{\pi d^2} = \frac{4*0.19*10^{-3}}{\pi*(0.05)^2} = 0.0968\,\frac{m}{s}$$

i) $R_e = \frac{Vd\rho}{\mu} = \frac{0.0968*0.05*800}{0.002} = 1936.0$

ii) $V_{x\,max} = 2V_{av.} = 2*0.0968 = 0.1936\,\frac{m}{s}$

iii) From Eq. 7.26

$$\frac{p_1 - p_2}{L} = 4\,V_{max}\frac{\mu}{R^2} = 32\mu\frac{V_{av}}{d^2}$$

$$\therefore\ p_1 - p_2 = \frac{32\mu V_{av}L}{d^2} = \frac{32*0.002*0.0968*500}{(0.05)^2} = 1239.04\,\frac{N}{m^2}$$

iv) $\frac{dp}{dL} = \frac{p_1 - p_2}{L} = \frac{1239.04}{500} = \frac{2.478\frac{N}{m^2}}{m} = 2.478\,pa/m$

v) $\tau_0 = \frac{(p_1 - p_2)d}{4L} = (1239.04)*\frac{0.05}{4*500} = 0.03098\,\frac{N}{m^2}, Eq.7.28$

7.5.3 Shear Stress in Horizontal Pipe.

A force balance for steady flow in horizontal pipe as in Fig. 7.5 yields

$$p_1(\pi r^2) - p_2(\pi r^2) - \tau(2\pi rL) = 0$$

$$or\ \tau = \frac{(p_1 - p_2)r}{2L} \qquad (7.27)$$

From Eq. 7.27

$at\ r = 0\ \tau = 0$

$r = R\quad \tau = \tau_0$

$$\tau_0 = \frac{(p_1 - p_2)d}{4L} \qquad (7.28)$$

Eq. 7.27 is valid for laminar & turbulent flow.

$\left(\frac{p_1 - p_2}{\rho g}\right)$ Represent the energy drop per unit weight (h_L) multiply Eq. 7.27 by (ρg/ρg) yields

$$\tau = \frac{\rho g r}{2L}\left(\frac{p_1 - p_2}{\rho g}\right) = \frac{\rho g h_L}{2L} r \qquad (7.29)$$

$$\therefore\ h_L = \frac{2\tau_0 L}{\rho g R} = \frac{4\tau_0 L}{\rho g d} \qquad (7.30)$$

$\tau = \tau_0\ at\ r = R$

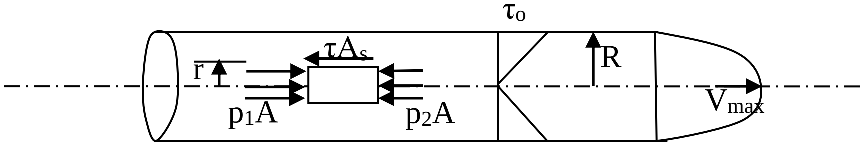
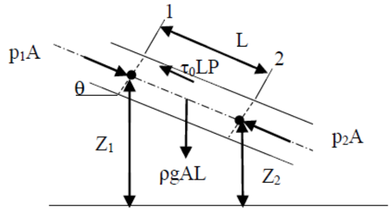


Figure 7.5: Forces on element in horizontal pipe.

7.5.4 Shear Stress in Inclined Pipe.

The energy equation may be written in pipe when related the loss to available energy reduction

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+\ z_2+h_{f_{1-2}}$$



Since the velocity head $\left(\frac{V^2}{2g}\right)$ is the same

$$h_f\ =\frac{p_1-p_2}{\rho g}+z_1-z_2 \qquad (7.31)$$

$$\therefore\ h_f=\frac{\Delta p}{\rho g}+\ \Delta z \qquad (7.32)$$

Applying the linear – momentum eqn. in the L-direction

$$\sum F_l=0=(p_1-p_2)A+\gamma AL\ sin\theta-\tau_0 LP=\dot{m}(V_2-V_1)=0$$

(P) is the wetted perimeter of the conduit ,i.e , the portion of the perimeter where the wall is in contact with the fluid when the conduit not circular pipe.

$$L\ sin\theta\ =\ z_1-z_2$$

$$\frac{p_1-p_2}{\rho g}+\ z_1-z_2=\frac{\tau_0 LP}{\rho gA} \qquad (7.33)$$

From Eq. 7.31& 7.33

$$h_f\ =\frac{\tau_0 LP}{\rho gA} \qquad (7.34)$$

From experiment

$$\tau_0=\lambda\frac{\rho}{2}\ V^2 \qquad (7.35)$$

$$\therefore\ h_f\ =\ \lambda\frac{\rho}{2}V^2\frac{LP}{\gamma A}=\ \lambda\frac{L}{R}\frac{V^2}{2g} \qquad (7.36)$$

R_h=A/P

R_h= hydraulic Radius of the conduit

For a pipe R_h=d/4 ; λ=f/4

Where λ is the non-dimensional factor, the h_f head loss due to friction can be written as follows,

$$\therefore\ h_f=\frac{f}{4}\frac{L}{d}\frac{4V^2}{2g}=f\frac{L}{d}\frac{V^2}{2g} \qquad (7.37)$$

Eq. 7.37 is the Darcy – Weisbach equation, valid for duct flows of any cross-section and for laminar and turbulent flow, ***f*** is the friction factor ***f***=4 λ

By equating Eq's 7.30 & 7.37

$\frac{4\tau_0 L}{\rho g d} = f\frac{L}{d}\frac{V^2}{2g}$

$\therefore\ \tau_0 = \frac{f\rho V^2}{8}$ (7.38)

In Hagen-Poiseuille eqn.

$V_{av} = \frac{\Delta p d^2}{32\mu L}$ From Eq. 7.26

$\Delta p = \rho g\, h_f ---\to\ h_f = \frac{\Delta p}{\rho g}$

$\therefore V_{av} = \frac{\rho g\, h_f d^2}{32\mu L}$

$h_f = \frac{32 V_{av}\mu L}{\rho g d^2} = f\frac{L}{d}\frac{V^2}{2g}$

$= \left(\frac{64 V_{av}\mu L}{2\rho g d^2}\right) = \frac{\frac{64}{\rho d V_{av}}}{\mu}\frac{L}{d}\frac{{V_{av}}^2}{2g} = \frac{64}{R_e}\frac{L}{d}\frac{{V_{av}}^2}{2g}$

$h_f = f\frac{L}{d}\frac{{V_{av}}^2}{2g} = \frac{64}{R_e}\frac{L}{d}\frac{{V_{av}}^2}{2g}$ (7.39)

$\therefore f = \frac{64}{Re}$ (7.40)

It applies to all roughness and may be used for the solution of laminar flow problems in pipes.

From above equations the laminar head loss as followes

$h_{f(laminar)} = \frac{64}{Re}\frac{L}{d}\frac{V_{av}^2}{2g} = \frac{32\mu L V_{av}}{\rho g d^2} = \frac{128\mu L Q}{\pi\rho g d^4}$ (7.41)

From Eq. 7.22

$p_1 - p_2 = \frac{4 V_{max}\mu L}{R^2} = \frac{32 V_{av}\mu L}{d^2}$

Pressure drop per unit weight

$h_f = \frac{\Delta p}{\rho g} = \frac{32\mu L V_{av}}{\rho g\, d^2}$ for laminar flow (7.42)

Ex.5

An oil of viscosity 0.9 Ns/m^2 and S.G. 0.9 is flowing through a horizontal pipe of 60 mm diameter. If the pressure drop in 100 m length of the pipe is $1800 kN/m^2$, determine:

(i) The rate of flow of oil.
(ii) The center-line velocity.
(iii) The total friction drags over 100 m length.
(iv) The power required to maintain the flow.
(v) The velocity gradient at the pipe wall.
(vi) the velocity and shear stress at 8mm from the wall

Sol.

Area of the pipe,

$A = \frac{\pi}{4} * (0,06)^2 = 2.827 * 10^{-3}(m^2)$ Pressure drop in (100m) length of the pipe, $\Delta p = 1800\ kN/m^2$

i) the rate of flow,Q

$p_1 - p_2 = \Delta p = \frac{32\mu V_{av} L}{d^2}$

$V_{av} = \frac{\Delta p\ d^2}{32\mu L}$

$\therefore\ V_{av} = \frac{1800*10^3*(0.06)^2}{32*0.9*100} = 2.25\ \frac{m}{s}$

Reynolds number, $Re\ = \frac{\rho V d}{\mu} = \frac{0.9*1000*2.25*\ 0.06}{0.9} = 135$

As Re is less than 2000, the flow is laminar and the rate of flow is,

$Q = A * V_{av} = 2.827 * 10^{-3} * 2.25 = 6.36 * 10^{-3}\frac{m^3}{S} = 6.36\frac{lit}{s}$

ii) the center-line velocity , V_{max}

$V_{max} = 2V_{av} = 2 * 2.25 = 4.5\frac{m}{s}$

iii) the total frictional drag over (100m) length

From $\tau_0 = \frac{(p_1 - p_2)d}{4L}$

$\therefore\ \tau_0 = 1800 * 10^3 * \frac{0.06}{4*100} = 270\ N/m^2$

$\therefore$ Friction drag for (100m) length

$F_d = \tau_0 * A_s = \tau_0 * \pi dL = 270 * \pi * 0.06 * 100$

$F_d = 5089\ N$

(iv) The power required to maintain the flow, P,

$P = F_d * V_{av} = 5089 * 2.25 = 11451\ W$

= 15.35 h.p

Alternatively,

$P = Q.\Delta p = 0.00636 * 1800 * 10^3 = 11448\ W$

(v) The velocity gradient at the pipe wall, $\left(\frac{du}{dy}\right)_{y=0}$;

$\tau_0 = \mu.\left(\frac{\partial u}{\partial y}\right)_{y=0}$

$or\ \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\tau_0}{\mu} = \frac{270}{0.9} = 300\ s^{-1}$

(vi) the velocity and shear stress at (8mm) from the wall,

$V = \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(1 - \frac{r^2}{R^2}\right)$

Or $V = -\frac{1}{4\mu}.\frac{\partial p}{\partial x}(R^2 - r^2)$

Here, $y = 8mm = 0.008m$

But y = R-r

$\therefore 0.008 = 0.03 - r ---\rightarrow r = 0.022m$

$\therefore\ V_{(8mm)} = +\frac{1}{4*0.9} * \frac{1800*10^3}{100}\ (0.03^2 - 0.022^2) = 2.08\frac{m}{s}$

For linear relation $\frac{\tau}{r} = \frac{\tau_0}{R} ---\rightarrow\ \tau_{(8mm)} = r * \frac{\tau_0}{R} = 0.022 * \frac{270}{0.03} = 198\ N/m^2$

Or $\tau = \frac{\Delta p}{2L} * r \quad from\ Eq. 7.27$

$\tau = 1800 * 10^3 * \frac{0.022}{2*100} = 198\ \frac{N}{m^2}$

Table 7.1: Summary of used equations in pipe

Velocity in circular pipe.	$V_x = \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left[1 - \frac{r^2}{R^2}\right]$
V_{max} (max. velocity)	$V_{max} = 2V_{av}$
V_{av} (Average velocity)	$V_{av} = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) = \frac{1}{2}V_{max}$
Pressure loss along pipe	$\frac{\Delta p}{L} = 4V_{max}\frac{\mu}{R^2} = \frac{32\mu V_{av}}{d^2}$
Wall shear stress	$\tau_0 = \frac{(p_1 - p_2)d}{4L}$
Shear stress at any ***r***	$\tau = \frac{(p_1 - p_2)r}{2L}$
Energy losses	$h_f = \frac{4\tau_0 L}{\rho g d}$
Energy loss by friction factor	$h_f = f\frac{L}{d}\frac{V^2}{2g}$
Hydraulic diameter	$d_h = \frac{4\ \ Area}{wetted\ primeter}$
Energy loss in Laminar flow	$h_{f\ laminar} = \frac{64}{R_e}\frac{L}{d}\frac{V_{av}^2}{2g} = \frac{32\mu LV}{\gamma d^2} = 128\mu LQ/\pi\rho g d^4$

Part-2
Turbulent Flow

7.6 Friction Factor Calculations.

Experimentation shows the following to be true in turbulent flow.

1- The head loss varies directly as the length of the pipe.
2- The head loss varies almost as the square of the velocity.
3- The head loss varies almost inversely as the diameter.
4- The head loss depends upon the surface roughness of the interior pipe wall.
5- The head loss depends upon the fluid properties of density and viscosity.
6- The head loss is independent of the pressure.

$h_f = f\frac{L}{d}.\frac{V^2}{2g}$

$f = f(V, d, \rho, \mu, \epsilon, \acute{\epsilon}, m)$

∈ is a measure of the size of the roughness projection and has the dimension of a length.

∈́ is a measure of the arrangement or spacing of the roughness elements.

m is a form factor.

For smooth $\in=\in' = m = 0 \rightarrow f = f(V, D, \rho, \mu)$ averaged into non-dimensionless group namely $\frac{\rho dV}{\mu} = Re$

For rough pipes the terms ∈,∈' may be made dimensionless by dividing by d

$\therefore f = F\left(\frac{\rho dV}{\mu}, \frac{\in}{d}, \frac{\in\prime}{d}, m\right)$ Proved by experimental plot of friction factor aganst the R_e on a log-log chart. Blasius presented his results by an empirical formula is valid up to about Re =100000

$f = \frac{0.316}{Re^{\frac{1}{4}}}$

In rough pipe ∈/d is called relative roughness.

$f = F\left(Re, \frac{\in}{d}\right)$ is limited and not permit variation of ∈'/d or m.

Moody has constructed one of the most convenient charts for determining friction factors. In laminar flow, the straight line masked “laminar flow” and the Hangen-Poiseuille equation is applied and from which $f = 64/Re$

$$h_f = f\frac{L}{d}\frac{V^2}{2g}\ ;\ V_{av} = \frac{\Delta p R^2}{8\mu L}$$

The Colebrook formula provides the shape of ∈/d = constant curves in the transient region

$$\frac{1}{\sqrt{f}} = -0.86\ln\left(\frac{\frac{\in}{d}}{3.7} + \frac{2.51}{Re\ \sqrt{f}}\right) \qquad (7.43)$$

7.7 Simple Pipe Problem.

Six variables enter into the problem for incompressible fluid, which are Q, L, d, h_f, V, ∈. Three of them are given (L, V, ∈) and three will be find. Now, the problems type can be solved as follows,

Problem	Given	To find (unknown)
I	$Q, L, d, V, \in$	h_F
II	$h_f, L, d, V, \in$	Q
III	$h_f, Q, L, V, \in$	d

In each of the above problem the following are used to find the unknown quantity

(i) The Darcy – Weisbach Equation.
(ii) The Continuity Equation.
(iii) The Moody diagram.

In place of the Moody diagram Fig. 7.6, the following explicit formula for *f* may be utilized with the restrictions placed on it

$$f = 0.0055\left[1 + \left(2000.\frac{\in}{d} + \frac{10^6}{Re}\right)^{\frac{1}{3}}\right] \text{ Moody equation}$$

$$4 * 10^3 \le Re \quad \le 10^7 \& \frac{\in}{D} \le 0.01$$

$$f = \frac{1.325}{\left[\ln\left(\frac{\in}{3.7d} + \frac{5.74}{Re^{0.9}}\right)\right]^2} \quad 10^{-6} \le \frac{\in}{D} \le 10^{-12} \ ,\ 5000 \le R_e \le 10^8 \qquad (7.44)$$

1% yield diff-with Darcy equation

The following formula can be used without Moody chart is

$$\frac{1}{f^{1/2}} \approx -1.8\log\left[\frac{6.9}{Re_d} + \left(\frac{\in/d}{3.7}\right)^{1.11}\right] \qquad (7.45)$$

Eq. 7.45 is given by Haaland which varies less than 2% from Moody chart.

7.7.1 Solution Procedures.

I- Solution for h_f.

With ***Q,∈,*** and ***d*** are known

$$Re \quad = \frac{Vd}{\upsilon} = \frac{4Q}{\pi d\upsilon}$$

And ***f*** may be looked up in Fig. 7.6 or calculated from Eq. 7.44. Substitution of (***f***) in Eq. 7.37 yields h_f the enrgy loss due to flow through the pipe per unit weight of fluid.

Ex.6

Determine the head (energy) loss for flow of 140 l/s of oil υ=0.00001 m^2/s through 400 m pipe length of 200 mm- diameter cost–iron pipe

Sol.

$Re \quad = \frac{4Q}{\pi D \upsilon} = \frac{4(0.14)}{\pi(0.2)(0.00001)} = \mathbf{89127}$

The relation roughness is ∈/D= 0.25/200= 0.00125 from a given diagram by interpolation ***f***= 0.023 by solution of Eq. 7.44, ***f***=0.0234; hence

$h_f = f \frac{L}{d}\frac{V^2}{2g} = 0.023 \frac{400}{0.2}\left[\frac{0.14}{\frac{\pi}{4}(0.2)^2}\right]^2 \frac{1}{2(9.81)}$

$h_f = 46.58m.$

II- Solution for Discharge Q.

V & f Are unknown then Darcy – Weisbach equation and moody diagram must be used simultaneously to find their values.

1- Givens $\begin{cases} \boldsymbol{\in/d} \\ \boldsymbol{f} \text{ value is assumed by inspection of the Moody diagram} \end{cases}$
2- Substitution of this trail ***f*** into the Darcy – equation produce a trial value of ***V.***
3- From $\boldsymbol{V}$ a trial $\boldsymbol{Re}$ is computed.
4- An improved value of ***f*** is found from moody diagram with help of $\boldsymbol{Re}$
5- When ***f*** has been found correct the corresponding $\boldsymbol{V}$ and $\boldsymbol{Q}$ is determined by multiplying by the area.

Ex.7

Water at 15 C° flow through a 300mm diameter riveted steel pipe, ∈=3mm with a head loss of 6 m in 300 m. Determine the flow rate in pipe.

Sol.

The relative roughness is ***∈/d*** = 0.003/0.3=0.01, and from diagram a trial ***f*** is taken as (0.038). By substituting into Eq. 7.37 Darcy equation

$6 = 0.038 \frac{300}{0.3} \frac{V^2}{2(9.81)}$

$\therefore \ V = 1.76 \frac{m}{s}$

At T=15C° → υ = $1.13 * 10^{-6} \frac{m^2}{s}$

$\therefore \ Re = \frac{Vd}{\upsilon} = \frac{1.715*0.3}{1.13*10^{-6}} = 467278$

From the Moody diagram ***f***=0.038 at $\left(Re \quad \& \ \frac{\in}{D}\right)$

And from Darcy → $V_{av} = \sqrt{\frac{h_f.d.2.g}{f.L}} = \sqrt{\frac{6*0.3*2*9.81}{0.038*300}} = 1.76 \frac{m}{s}$

$$\therefore Q = AV = \pi\,(0.15)^2 \sqrt{\frac{(6*0.3)(2)(9.81)}{(0.038)(300)}} = 0.1245\,\frac{m^3}{s}$$

III- Solution for Diameter d.

Three unknown in Darcy-equation ***f***, ***V***, ***d***, two in the continuity equation ***V, d*** and three in the ***Re*** number equation

To element the velocity in Eq. 7.37 & in the expression for ***Re***, simplifies the problem as follows.

$$h_f = f\,\frac{L}{d}\,\frac{Q^2}{2g\left(\frac{d^2\pi}{4}\right)^2}$$

$$Or \quad d^5 = \frac{8LQ^2}{h_f g\pi^2}\,f = C_1\,f \qquad (7.46)$$

In which C_1 is the known quantity $\frac{8LQ^2}{h_f g\pi^2}$

From continuity $\quad Vd^2 = \frac{4Q}{\pi}$

$$Re = \frac{Vd}{\upsilon} = \frac{4Q}{\pi\upsilon}\frac{1}{d} = \frac{C_2}{d} \qquad (7.47)$$

C_2 is the known quantity $\frac{4Q}{\pi\upsilon}$ the solution is now effected by the following procedure

1- Assume the value of ***f***.
2- Solve Eq. 7.46 for ***d***.
3- Solve Eq. 7.47 for ***Re.***
4- Find the relative roughness ***∈/d***.
5- With R_e and ***∈/d***, Look up new ***f*** from a diagram.
6- Use the new ***f***, and repeat the procedure.
7- When the value of ***f*** does not change in the two significant steps, all equations are satisfied and the problem is solved.

Ex.8

Determine the size of clean wrought-iron pipe required to convey 4000 gpm oil, υ=0.0001 $\frac{ft^2}{s}$, 10000 ft pipe length with a head loss of 75 ft .lb/lb.

Sol.

The discharge is $Q = \frac{4000}{448.4} = 8.93\ cfs$

From Eq. 7.46, $d^5 = \frac{8LQ^2}{h_f g\pi^2}\,f = \frac{8*10000*8.93^2}{75*32.2*\pi^2}\,f = \mathbf{267.65}\,\boldsymbol{f}$

And from Eq. 7.47,

$$Re = \frac{4Q}{\pi\upsilon}\frac{1}{d} = \frac{4*8.93}{\pi*0.0001}\frac{1}{d} = \frac{113700}{d}$$

And from Table 7.2 ∈= 0.00015 ft

If *f*=0.02 (assumed value) , ∴ d = 1.35 ft

$$\left.\begin{array}{l} Re = 81400 \\ \in/d = 0.00011 \end{array}\right\} \quad \text{from Moody chart } f = 0.0191$$

In repeating the procedure, d = 1.37 ft → $Re = 82991 --\rightarrow f = 0.019$
Therefore
$d = 1.382 * 12 = 16.6\ in$

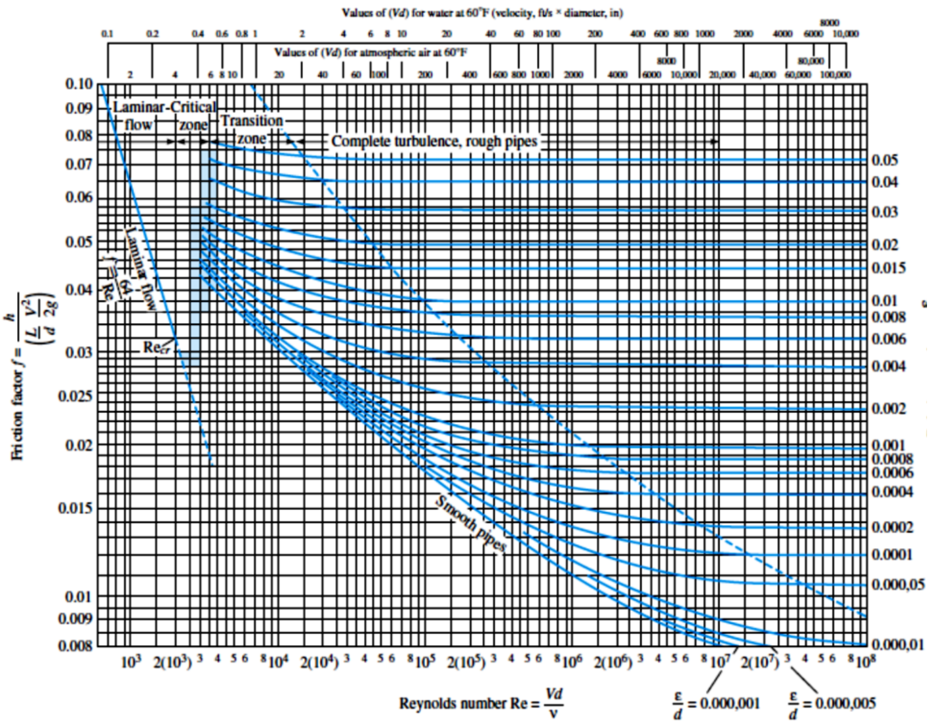


Figure 7.6: The Moody chart for pipe friction with smooth and rough walls [1].

Table 7.2: Recommended roughness values [1].

Material	Condition	ϵ ft	ϵ mm	Uncertainty, %
Steel	Sheet metal, new	0.00016	0.05	± 60
	Stainless, new	0.000007	0.002	± 50
	Commercial, new	0.00015	0.046	± 30
	Riveted	0.01	3.0	± 70
	Rusted	0.007	2.0	± 50
Iron	Cast, new	0.00085	0.26	± 50
	Wrought, new	0.00015	0.046	± 20
	Galvanized, new	0.0005	0.15	± 40
	Asphalted cast	0.0004	0.12	± 50
Brass	Drawn, new	0.000007	0.002	± 50
Plastic	Drawn tubing	0.000005	0.0015	± 60
Glass	—	Smooth	Smooth	
Concrete	Smoothed	0.00013	0.04	± 60
	Rough	0.007	2.0	± 50
Rubber	Smoothed	0.000033	0.01	± 60
Wood	Stave	0.0016	0.5	± 40

7.7.2 General Applications

Application-1

A pump delivers water from a tank (A)(water surface elevation=110m) to tank B (water surface elevation= 170m). The suction pipe is 45m long and 35cm in diameter the delivered pipe is 950m long 25cm in diameter. Loss head due to friction $\boldsymbol{h_{f1}} = 5m$ and $\boldsymbol{h_{f2}}$ = 3m If the piping are from

pipe(1)= steel sheet metal

pipe(2)= stainless – steel

Calculate the following

i) The discharge in the pipeline

ii) The power delivered by the pump.

Sol.

Given

$\nu_w = 1.007 * 10^{-6} \frac{m^2}{s}$

$d_1 = 35\ cm = 0.35m\ ;\ d_2 = 25cm = 0.25m$

$L_1 = 45m\ ;\ L_2 = 950\ m$

From table 7.2 $\in_1 = 0.05\ mm$

$\in_2 = 0.002\ mm$

$\frac{\in_1}{d_1} = \frac{0.05}{350} = 1.428 * 10^{-4}$

$\frac{\in_2}{d_2} = \frac{0.002}{250} = 8 * 10^{-6}$

Assume $f_1 = 0.013;\ f_2 = 0.008$

$h_{f1} = f_1 \frac{L_1}{d_1} \cdot \frac{V_1^2}{2g}$

$5 = 0.013 \frac{45}{0.35} \cdot \frac{V_1^2}{2*9.81} ---\rightarrow V_1 = 7.66 \frac{m}{s} --\rightarrow Re_1 = \frac{V\ d}{\nu} = \frac{7.66*0.35}{1.007*10^{-6}}$

$Re_1 = 2662363 = 2.66 * 10^6$

$h_{f2} = f_2 \frac{L_2}{d_2} \frac{V_2^2}{2g} = 0.008 \frac{950}{0.25} \cdot \frac{V_2^2}{2*9.81} = 3.0m$

$V_2 = 1.39 \frac{m}{s} \quad Re_2 = \frac{1.39*0.25}{1.007*10^{-6}} = 3.45 * 10^5$

1st Trail

$\left(Re_1 \& \frac{\in_1}{d_1}\right) --\rightarrow f_1 = 0.0138$

$\left(Re_2 \& \frac{\in_2}{d_2}\right) -\rightarrow f_2 = 0.014$

$h_{f1} = 5 = 0.0138 \frac{45}{0.35} \cdot \frac{V_1^2}{2*9.81} -\rightarrow V_1 = 7.435 \frac{m}{s} ---\rightarrow Re_1 = 2.58 * 10^6$

$h_{f2} = 3 = 0.014 \frac{950}{0.25} \frac{V_2^2}{2*9.81} --\rightarrow V_2 = 1.051 \frac{m}{s} --\rightarrow Re_2 = 2.6 * 10^5$

$2^{\ nd}\ trial$

$\left(Re_1 \& \frac{\in_1}{d_1}\right) \quad f_1 = 0.0165\ , f_2 = 0.015$

From ***f₁ &f₂***

$h_{f1} = 5 = 0.0165 \frac{45}{0.35} \frac{V_1^2}{2*9.81} \rightarrow V_1 = 6.8\, m/s$

$h_{f2} = 3 = 0.015 \frac{950}{0.25} \frac{V_2^2}{2*9.81} \rightarrow V_2 = 1.01\, m/s$

$Re_1 = 2.36*10^6$
$Re_2 = 2.52*10^5$

$3^{rd}\ \ trial$

$\left(Re_1 \& \frac{\epsilon_1}{d_1}\right),\ \left(Re_2 \& \frac{\epsilon_2}{d_2}\right) \ \rightarrow\ f_1 = 0.0169, f_2 = 0.015$

From Darcy-equation gives V_1=0.6.72 m/s, V_2=1.016 m/s.
$Q = A_1 * V_1 =$ **0.6462** m^3/s
From energy equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_1^2}{2g} + z_2 + h_f$$

$\frac{(6.72)^2}{2*9.81} + 110 + h_p = \frac{(1.016)^2}{2*9.81} + 170 + 8$ Since $p_1 = p_2$

$\boldsymbol{h_p}$ = 65.75 m

$P = \gamma Q h_p = 9810 * 0.6462 * 65.75 = 416.8\, kW$ The power delivered by the pump.

Application-2

In a pipeline of diameter 350mm and length 75m, water is flowing at a velocity of 2.8 m/s. Find the head lost due to friction, using Darcy-Eq.& Moody chart, pipe material is Steel–Riveted kinematic viscosity υ= 0.012 stoke

Sol.

$h_f = f \frac{L}{d} . \frac{V^2}{2g}\ ; d = 0.35m\ , L = 75m; V = 2.8 \frac{m}{s}$

From table 7.2 for steel riveted ϵ=3.0 mm

$\frac{\epsilon}{d} = \frac{0.003}{0.35} = 8.57 * 10^{-3}$

$1 \frac{m^2}{s} = 10^4\ stoke\ \therefore \upsilon = \ 0.012 * 10^{-4} \frac{m^2}{s}$

$Re \ = \frac{Vd}{\upsilon} = \frac{2.8*0.35}{0.012*10^{-4}} = 816666 = 8.1 * 10^5$

$at\ \left(Re\ \ \&\ \frac{\epsilon}{d}\right) \longrightarrow f = 0.0358$

$\therefore\ h_f = 0.0358 \frac{75}{0.35} \frac{2.8^2}{2*9.81} = 3.0m$

By determine the value of ***f*** by Eq. 7.45.

$$\frac{1}{f^{\frac{1}{2}}} \approx -1.8 \log\left[\frac{6.9}{Re_d} + \left(\frac{\frac{\epsilon}{d}}{3.7}\right)^{1.11}\right]$$

$\frac{1}{f^{\frac{1}{2}}} = -1.8\log(\frac{6.9}{8.16*10^6} + (\frac{8.57*10^{-3}}{3.7})^{1.11}) = 5.2646$

f=0.036 ***Δf*** = 0.0002

Application-3

Oil having absolute viscosity 0.1 Pa.s and relative density 0.85 flow through an iron pipe with diameter 305mm and length 3048 m with flow rate $44.4 * 10^{-3}\frac{m^3}{s}$. Determine the head loss per unit weight in pipe.

Sol.

$V = \frac{Q}{A} = \frac{44.4*10^{-3}}{\frac{1}{4}\pi(0.305)^2} = 0.61\frac{m}{s}$

$Re = \frac{Vd\rho}{\mu} = \frac{0.61*0.305*850}{0.1} = 1580$

i.e the flow is laminar .

$f = \frac{64}{R_e} = \frac{64}{1580} = 0.0407$

$\therefore\ h_f = f\frac{L}{d}\frac{V}{2g} = 0.0407 * \frac{3048}{0.305} * \frac{(0.61)^2}{2g} = 7.71\ m$

7.8 Minor Losses.

The losses which occur in pipelines because of bend, elbows, joints, valves, etc, are called minor losses $\boldsymbol{h_m}$. the total losses in pipeline are

$h_L = h_f + h_m$ (7.48)

Also, others minor losses can be explain as follows,

A. Losses due to sudden expansion in pipe.

From momentum equation

$$\sum Fx = \rho Q\ (V_2 - V_1)$$

$Q = A_2V_2$

$p_1A_1 - p_2A_2 = \rho A_2(V_2^2 - V_1V_2)$

Divided by γA_2 since $A_1 = A_2$

$\frac{p_1-p_2}{\gamma} = \frac{V_2^2-V_1V_2}{g}$ $---(a)$

From Bermonlli's equation between section 1&2 as in Fig.7.7

$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_m$

$\frac{p_1-p_2}{\gamma} = \frac{V_2^2-V_1^2}{2g} + h_m$ $---(b)$

Equating Eq. (a & b)

$\frac{V_2^2-V_1V_2}{g} = \frac{V_2^2-V_1^2}{2g} + h_m$

$h_m = \frac{V_2^2-V_1V_2}{g} - \frac{V_2^2-V_1^2}{2g} = \frac{2V_2^2-2V_1\,V_2-V_2^2+V_1^2}{2g}$

$$h_m = \frac{V_1^2 - 2V_1V_2 + V_2^2}{2g} = \frac{(V_1 - V_2)^2}{2g} \qquad ---(c)$$

$$Q = A_1V_1 = A_2V_2$$

$$V_2 = \frac{A_1}{A_2}V_1 \qquad substitute\ in\ Eq.(c)$$

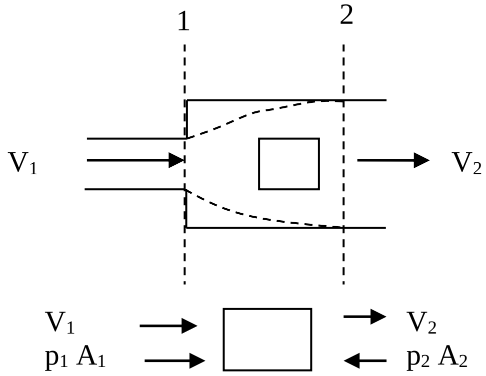


Figure 7.7: Sudden expansion.

$$\therefore\ h_m = \frac{V_1^2 - 2V_1\frac{A_1}{A_2}V_1 + \left(\frac{A_1}{A_2}V_1\right)^2}{2g}$$

$$h_m = \frac{V_1^2\left(1 - 2\frac{A_1}{A_2} + \left(\frac{A_1}{A_2}\right)^2\right)}{2g}$$

$$h_m = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 = K\frac{V_1^2}{2g} \qquad (7.49)$$

$$K = \left(1 - \frac{A_1}{A_2}\right)^2 = \left(1 - \frac{d_1^2}{d_2^2}\right)^2$$

If sudden expansion from pipe to a reservoir $\frac{d_1}{d_2} = 0$

$$\therefore\ h_m = \frac{V_1^2}{2g}$$

B. Head loss due to a sudden contraction in the pipe cross section.

$$Q = A_2V_2 = A_cV_c$$

$$C_c = A_c/A_2$$

$$p_cA_2 - p_2A_2 = \rho Q(V_2 - V_c)$$

$$p_cA_2 - p_2A_2 = \rho A_2(V_2^2 - V_2V_c)$$

Divided by $\gamma\ A_2$

$$\frac{p_c - p_2}{\gamma} = \frac{V_2^2 - V_cV_2}{g} \qquad ---(a)$$

Applying B.E between sections c & 2

$$\frac{p_c}{\gamma} + \frac{V_c^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_m$$

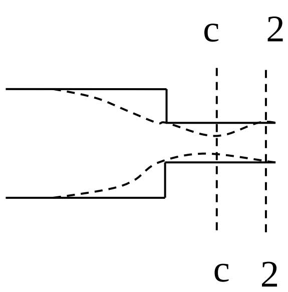


Figure 7.8: Sudden contraction.

$$\frac{p_c - p_2}{\gamma} = \frac{V_2^2 - V_c^2}{2g} + h_m - - - - (b)$$

$V_c = \frac{A_2}{A_C} V_2$, Equating Eq (a& b)

$$\frac{V_2^2 - V_c V_2}{g} = \frac{V_2^2 - V_C^2}{2g} + h_m$$

$$h_m = \frac{V_2^2 - V_C V_2}{g} - \left(\frac{V_2^2 - V_C^2}{2g}\right) = \frac{2V_2^2 - 2V_c V_2 - V_2^2 + V_C^2}{2g}$$

$$h_m = \frac{V_C^2 - 2V_c V_2 + V_2^2}{2g} = \frac{V_2^2}{2g}\left(\left(\frac{A_2}{A_c}\right)^2 - \frac{2A_2}{A_c} + 1\right)$$

$$h_m = \frac{V_2^2}{2g}\left[\left(\frac{1}{C_c}\right)^2 - 1\right]^2$$

$$h_m = K\frac{V_2^2}{2g} \qquad (7.50)$$

$K = \left(\frac{1}{C_c} - 1\right)^2$, From experimental, $K \approx 0.42\left(1 - \frac{d_2^2}{d_1^2}\right)$

A_c is the cross – sectional area of the vena-contracts

C_c is the coefficient of contraction is defined by $C_c = \frac{A_c}{A_2}$

C. ***<u>The head loss at the entrance to a pipe line.</u>***

From a reservoir the head losses is usually taken as

$h_m = \frac{0.5V^2}{2g}$ is the opening is square-edged

$h_m = \frac{0.01V^2}{2g} \sim \frac{0.05V^2}{2g}$ if the rounded entrance

Table 7.3 lists the loss coefficient ***K*** for four types of valve, three angles of elbow fitting and two tee connections. Fitting may be connected by either internal screws or flanges, hence the two are listings.

Table 7.3: Head loss coefficients K for typical fittings.

	Nominal diameter, in								
	Screwed				Flanged				
	$\frac{1}{2}$	1	2	4	1	2	4	8	20
Valves (fully open):									
Globe	14	8.2	6.9	5.7	13	8.5	6.0	5.8	5.5
Gate	0.30	0.24	0.16	0.11	0.80	0.35	0.16	0.07	0.03
Swing check	5.1	2.9	2.1	2.0	2.0	2.0	2.0	2.0	2.0
Angle	9.0	4.7	2.0	1.0	4.5	2.4	2.0	2.0	2.0
Elbows:									
45° regular	0.39	0.32	0.30	0.29					
45° long radius					0.21	0.20	0.19	0.16	0.14
90° regular	2.0	1.5	0.95	0.64	0.50	0.39	0.30	0.26	0.21
90° long radius	1.0	0.72	0.41	0.23	0.40	0.30	0.19	0.15	0.10
180° regular	2.0	1.5	0.95	0.64	0.41	0.35	0.30	0.25	0.20
180° long radius					0.40	0.30	0.21	0.15	0.10
Tees:									
Line flow	0.90	0.90	0.90	0.90	0.24	0.19	0.14	0.10	0.07
Branch flow	2.4	1.8	1.4	1.1	1.0	0.80	0.64	0.58	0.41

Entrance losses are highly dependent upon entrance geometry, but exit losses are not. Sharp edges or protrusions in the entrance cause large zones of flow separation and large losses as shown in Fig. 7.9. As in Fig. 7.10, a bend or curve in a pipe, always induces a loss larger than the simple Moody friction loss, due to flow separation at the walls and a swirling secondary flow arising from centripetal acceleration.

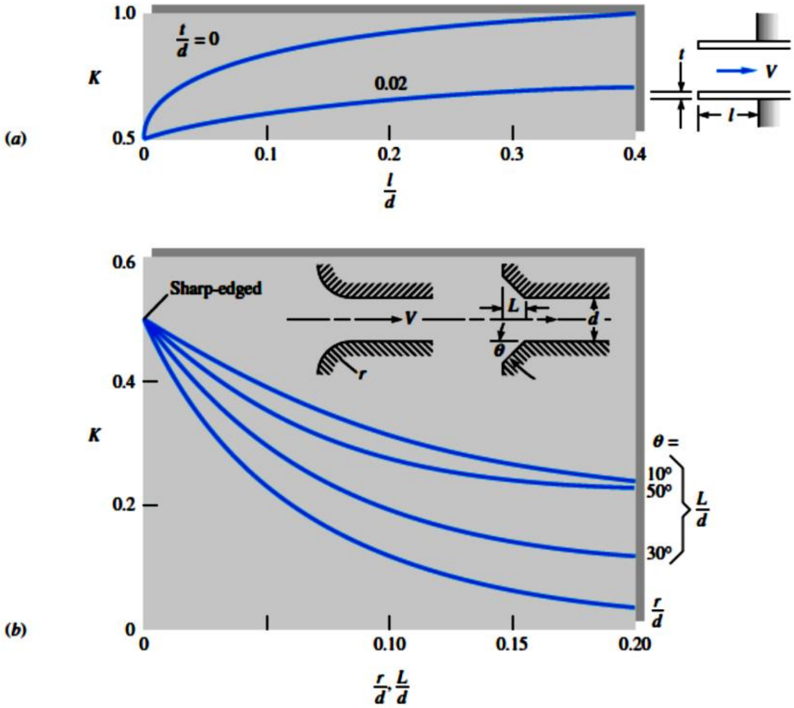


Figure 7.9: Entrance and exit loss coefficients,(a) reentrant inlets, (b) rounded and beveled inlets. Exit losses are K=1.

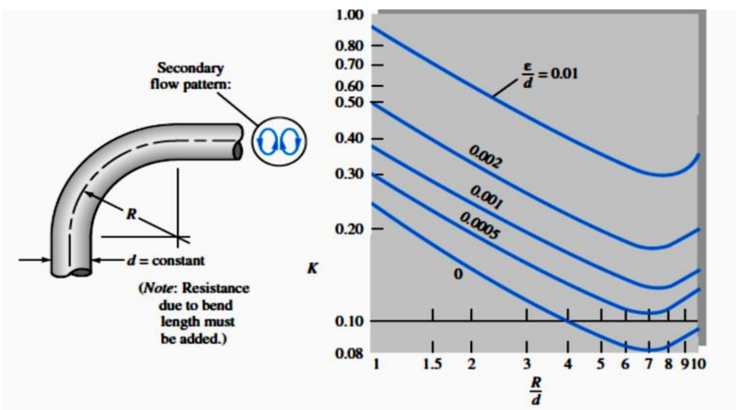


Figure 7.10: Resistance coefficients for 90° bends.

Table 7.3 gives the losses coefficients for the fully open condition. In case of partially open valve the losses can be much higher. Fig. 7.11 gives average losses for three valves as a function of percentage open. The opining distance ratio ***h***/D as the x-axis in Fig. 7.11 is shown by Fig.7.12 of valve geometry.

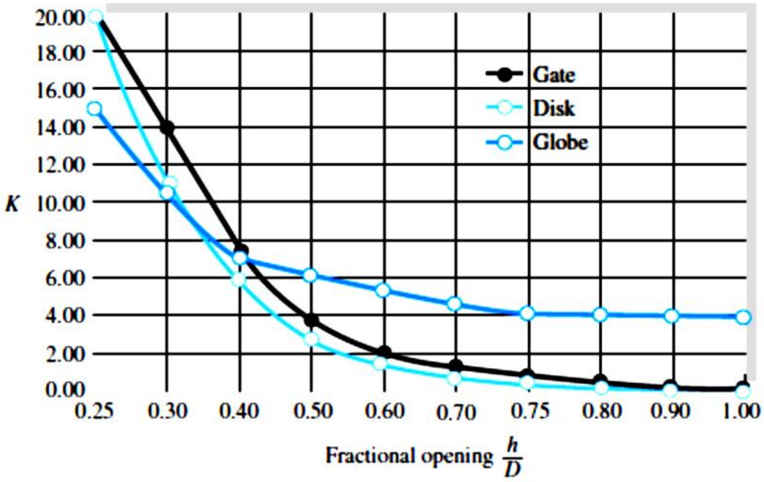


Figure 7.11: Average-loss coefficients for partially open valves.

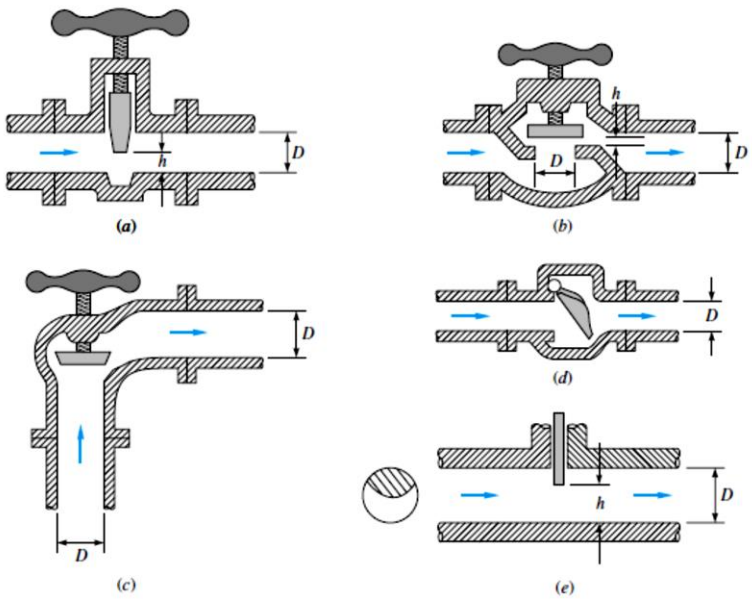


Figure 7.12: Typical commercial valve; (a) gate valve, (b) globe valve, (c) angle valve, (d) swing-check valve, (e) disk-type gate valve.

Application-4

Water, ρ = 1.94 slugs/ft^3, $and\ \nu = 1.1 * 10^{-5} \frac{ft^2}{s}$, is pumped between two reservoir at $0.2 \frac{ft^3}{s}$ through 400 ft of 2 in diameter pipe and several minor losses, as shown in figure. The roughness ratio is $\frac{\epsilon}{d} = 0.001$. Compute the horse power required.

Sol.

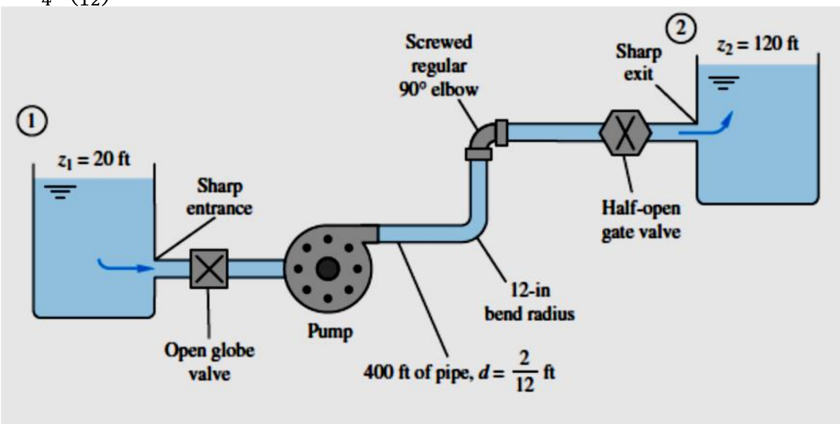
Write the steady- flow energy equation between section 1 &2 the two reservoir surface:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \left(\frac{p_2}{2g} + \frac{V_2^2}{2g} + z_2\right) + h_f + \sum h_m - h_p$$

Where h_p is the head increase across the pump $p_1 = p_2$, $V_1 = V_2 \approx 0$, solve for the pump head

$$h_p = z_2 - z_1 + h_f + \sum h_m = 120 - 20 + \frac{V^2}{2g}\left(f\frac{L}{d} + \sum K\right)$$

$$V = \frac{Q}{A} = \frac{0.2}{\frac{1}{4}\pi\left(\frac{2}{12}\right)^2} = 9.17\frac{ft}{s}$$



Now list and sum the minor loss coefficients

Loss	K
Sharp entrance (fig, 7.9)	0.5
Open globe value (2 in Table 7.3)	6.9
12-in bend Fig. 7.10 $\frac{R}{d} = 6, \frac{\epsilon}{d} = 0.001$	0.15
Regular 90 elbow ($Table$ 7.3)	0.95
Half – closed gate value (Fig. 7.11)	3.8
Sharp exit (Fig. 7.9)	1.0
$\sum K$	13.3

Calculate the Re and pipe friction factor

$Re \; = \frac{Vd}{\nu} = \frac{9.17\left(\frac{2}{12}\right)}{1.1*10^{-5}} = 139000$

For $\frac{\epsilon}{d} = 0.001$, from the Moody chart read f = 0.0216

$\therefore \; h_p = 100 + \frac{9.17^2}{2(32.2)}\left[\frac{0.0216(400)}{\left(\frac{2}{12}\right)} + 13.3\right]$

$h_p = 100 + 84 = 184 \; ft \; pump \; head$

$P = \rho g Q h_p = 1.94(32.3)(0.2)(184) = 2300 \; \frac{ft \; lb_f}{s}$

$1 \quad h.p. = 550 \; \frac{ft \; lb_f}{s} \;\; \therefore P = \frac{2300}{550} = 4.2 \; h.p.$

For an efficiency 70 to 80% , a pump is needed with an input power about 6 h.p.

Application-5

Water is to be supplied to the inhabitants of a college campus through a supply main. The following data is given,

Distance of the reservoir from the campus = 3000 m

Number of inhabitance = 4000

Consumption of water per day of each inhabitant= 180 liters

Loss of head due to friction = 18 m,

Co-efficient of friction for the pipe, f = 0.007. If the half of the daily supply is pumped in 8 hr, determine the size of the supply main (***d***) .

Sol.

Total supply per day $= 4000 * \frac{180}{1000} = 720 \frac{m^3}{day}$

$Q = \frac{720}{2*8*3600} = 0.0125 \frac{m^3}{s}$

$h_f = 18m$

Assume f= 0.03 $\qquad \therefore d^5 = \frac{8*L\,Q^2}{h_f \; g \; \pi^2} * f = \frac{8*3000*(0.0125)^2}{18*9.81*\pi^2} * 0.03 = 6.455 * 10^{-5}$

$\therefore d = 0.1452m$

$R_e = \frac{4Q}{\pi \nu d} = \frac{4*0.0125}{\pi*1.007*10^{-6}*0.1452} = 108904 = 1.08 * 10^5$

$\frac{\epsilon}{d} = \frac{0.046*10^{-3}}{0.1452} = 3.168 * 10^{-4}$ $\qquad$ From $\left(R_e \; \& \; \frac{\epsilon}{d}\right) - - \rightarrow f = 0.0195$

$d^5 = 4.2 * 10^{-5} - - \rightarrow d = 0.13352 \; m$.

Minor losses may be expressed in terms of the equivalent $\boldsymbol{L_e}$ of pipe that has the same head loss in m.N /N

$f \frac{L_e}{d}\frac{V^2}{2g} = K\frac{V^2}{2g}$

K is the sum of several losses, solving for L_e gives $L_e = \frac{Kd}{f}$ (7.51)

Application-6

I) Find the discharge through the pipeline as in below figure for H=10m, II) determine the head loss $\boldsymbol{h_L}$ for Q=60 l/s. III) compare the result of discharge with equivalent length .

Sol.

The energy equation applied between points 1 & 2, including all the losses, may be written

I) $\mathrm{H}_1 + 0 + 0 = \frac{V_2^2}{2g} + 0 + 0 + \frac{1}{2}\frac{V_2^2}{2g} + f\frac{102}{0.2032}\frac{V_2^2}{2g} + 2 * 0.26\frac{V_2^2}{2g} + \frac{5.3\,V_2^2}{2g}$

Loss coefficients (K):-

Entrance = 0.5

Each elbow = 0.26

Globe valve (partially open h/d=0.6) = 5.3

$\therefore\ H_1 = \frac{V_2^2}{2g}\,(7.32 + 502\,f) \qquad ------(A)$

When the head is given, this problem is solved as the second type of simple pipe problem. If $\frac{\in}{d} = \frac{0.26}{203.2} = 1.28 * 10^{-3}$, f = 0.0205

$10 = \frac{V_2^2}{2g}\,(\,7.32 + 502 * 0.0205) --\rightarrow\ V_2 = 3.337\ \frac{m}{s}$

$\nu = 1.01 * 10^{-6}\frac{m^2}{s}$

$\frac{\in}{d} = 0.00128\ ;\ R_e = \frac{(3.337*0.2032)}{(1.01*10^{-6})} = 6.7 * 10^5$

From Moody chart at $\left\{Re \quad \& \ \frac{\in}{d}\right\} --\rightarrow f = 0.0208$

Repeating the procedure gives $V_2 = 3.32\frac{m}{s}\ , Re \quad = 6.6 * 10^5\ , and\ \frac{\in}{d},\ f =$ 0.0209. from eq. A gives V_2=3.31 m/s. The discharge is

$Q = \ V_2 A_2 \ = (3.31)\left(\frac{\pi}{4}\right)(0.2032)^2 = 107.34\ l/s$

II) For the second part , with Q is known, the solution is straight forward;

$V_2 = \frac{Q}{A} = \frac{0.06}{\left(\frac{\pi}{4}\right)(0.2032)^2} = \ 1.85\ \frac{m}{s} \quad Re \quad = 3.7 * 10^5\ and\ \frac{\in}{d}, f = 0.0212$

From Eq. A

$\therefore\ h_L = \frac{(1.85)^2}{2(9.806)}(6.32 + 502 * 0.0212) = 2.959\ m$

III) With equivalent lengths Eq. 7.51 the value of $\boldsymbol{f}$ is an approximated, say $\boldsymbol{f}$ = 0.0205. The sum of minor losses is K= 6.32

$L_e = \frac{Kd}{f} = \frac{6.32*0.2032}{0.0205} = 62.64\ m$

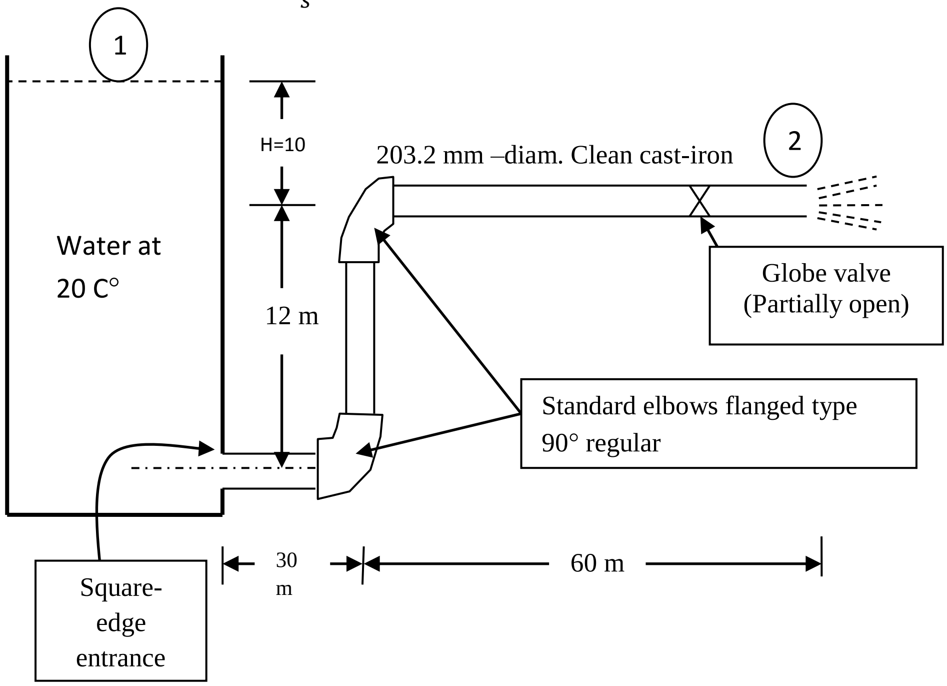
The total length of pipe is

$62.64 + 102 = 164.64\ m$

By Darcy equation. $10 = f\ \frac{L+L_e}{d}\frac{V_2^2}{2g} = f\frac{164.64}{0.2032}\frac{V_2^2}{2g}$

If $f = 0.0205, V_2 = 3.43\frac{m}{s},\ Re = 6.9 * 10^5\ and\frac{\in}{d}, f = 0.0203$

From eq.A , $V_2 = 3.347\,\frac{m}{s}\, and\ Q = 108.5\ l/s$



7.9 Pipe in Series.

In this typical series–pipe system as in Fig. 7.13, the H (head) is required for a given Q or the Q wanted for a given H. Applying the energy equation from A to B including all losses gives

$$H + 0 + 0 = 0 + 0 + 0 + K_e\frac{V_1^2}{2g} + f_1\frac{L_1}{d_1}\frac{V_1^2}{2g} + \frac{(V_1-V_2)^2}{2g} + f_2\frac{L_2}{d_2}\frac{V_2^2}{2g} + \frac{V_2^2}{2g} \quad (7.52)$$

From continuity eqn.

$$V_1 d_1^2 = V_2 d_2^2$$

V_2 is eliminated from eqn , so that

$$H = \frac{V_1^2}{2g}\left\{K_e + \frac{f_1 L_1}{d_1} + \left[1 - \left(\frac{d_1}{d_2}\right)^2\right]^2 + \frac{f_2 L_2}{d_2}\left(\frac{d_1}{d_2}\right)^4 + \left(\frac{d_1}{d_2}\right)^4\right\} \quad (7.53)$$

For known lengths and sizes of pipes this reduces to

$$H = \frac{V_1^2}{2g}\,(C_1 + C_2 f_1 + C_3 f_2) \quad (7.54)$$

$C_1, C_2\ \&\ C_3\ are\ Known$

- Q is given then R_e is computed and $f's$ may be looked up in Moody diagram then H is found by direct substitution.
- For a given H, the values of V_1, f_1, f_2 are unknown in Eq. 7.54.

By Assuming values of $f_1\&\, f_2$ (may be equaled) then $V_1\ is\ found\ 1^{st}\ trial$ and from $V_1 \dashrightarrow \boldsymbol{Re's}$ are determined and values of $f_1\&\, f_2$ look up from Moody diagram.

And at these value , a better V_1 is computed from Eq. 7.53 since ***f*** varies so slightly with Re the trial solution converges very rapidly. The same procedures apply for more than two pieces in series.

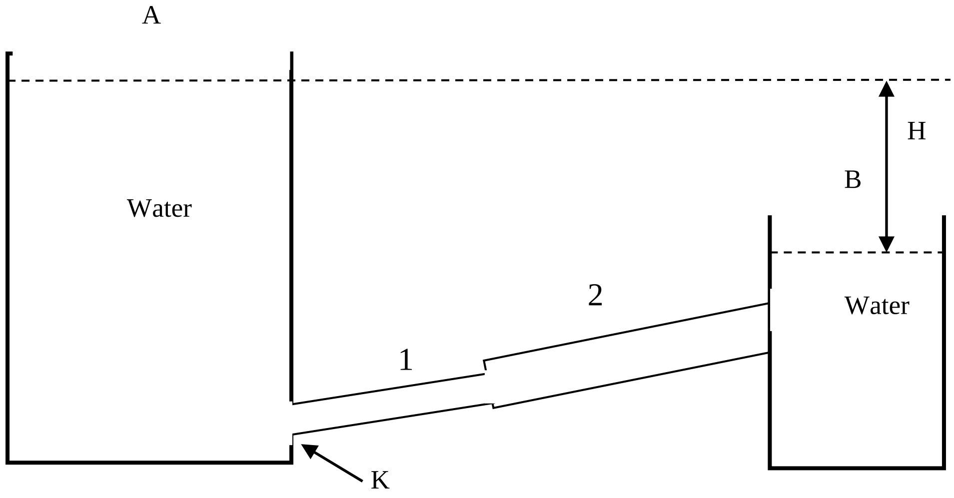


Figure 7.13: Series pipe.

7.10 Equivalent pipes.

Two pipe system (in series) are said to be equivalent when the same head loss produces the same discharge in both system. From Darcy equation.

$$h_{f1} = f_1 \frac{L_1}{d_1} \frac{Q_1^2}{\left(d_1^2 \frac{\pi}{4}\right)^2 2g} = f_1 \frac{L_1}{d_1^5} \frac{8Q_1^2}{\pi^2 g}$$

And for a second pipe

$$h_{f2} = f_2 \frac{L_2}{d_2^5} \frac{8Q_2^2}{\pi^2 g}$$

For the two pipes to be equivalent

$$h_{f1} = h_{f2} \qquad Q_1 = Q_2$$

After equating $h_{f1} = h_{f2}$ and simplifying

$$\frac{f_1 L_1}{d_1^5} = \frac{f_2 L_2}{d_2^5}$$

Solving for L_2 gives

$$L_2 = L_1 \frac{f_1}{f_2} \left(\frac{d_2}{d_1}\right)^5 \qquad (7.55)$$

Ex.9

From Fig. 7.13, $K_e = 0.5, L_1 = 300m , d_1 = 600mm, \in_1 = 2mm, L_2 = 240m, d_2 = 1m, \in_2 = 0.3mm , \nu = 3 * 10^{-6} \frac{m^2}{s}$ $and\ H = 6m.$ Determine the discharge through the system.

Sol.

From the energy Eq. 7.53

$$6 = \frac{V_1^2}{2g}\left[0.5 + f_1\frac{300}{0.6} + (1-0.6^2)^2 + f_2\frac{240}{1.0}\,0.6^4 + 0.6^4\right]$$

After simplifying

$$6 = \frac{V_1^2}{2g}\,(1.0392 + 500f_1 + 31.104f_2)$$

From $\frac{\epsilon_1}{d_1} = 0.0033, \frac{\epsilon_2}{d_2} = 0.0003,$ and Moody diagram values of f 's are assumed for the fully turbulent range.

$f_1 = 0.026 \quad f_2 = 0.015$

By solving for V_1 with these value , $V_1 = 2.848\frac{m}{s}, V_2 = 1.025\frac{m}{s}$

$$Re_1 = \frac{2.848*0.6}{3*10^{-6}} = 569600$$

$$Re_2 = \frac{1.025*1.0}{3*10^{-6}} = 341667$$

At these ***Re's*** and from Moody diagram, $f_1 = 0.0265, f_2 = 0.0168,$ by solving again for $V_1, V_1 = 2.819\frac{m}{s}$ and $Q = 0.797\frac{m^3}{s}$

Ex.10

Solve ***Ex.9*** by means of equivalent pipes.

Sol.

First by expressing the minor losses in terms of equivalent length, for pipe 1, since $K=K_e+(1-(d1/d2)^2)^2$

$$K_1 = 0.5 + (1-0.6^2)^2 = 0.91 ----\rightarrow\ L_{e1} = \frac{K_1 d_1}{f_1} = \frac{0.91*0.6}{0.026} = 21m$$

For pipe-2 $K_2 = 1 ----\rightarrow\ L_{e2} = \frac{K_2 d_2}{f_2} = \frac{1*1}{0.015} = 66.7\,m$

After selecting f_1 & f_2 . The problem is reduced to 321m of 600-mm diam. & 306.7 m of 1-m pipe diam.

By expressing the 1-m pipe terms of an equivalent length of 600-mm pipe, by Eq. 7.55

$$L_e = \frac{f_2}{f_1}\,L_2\left(\frac{d_1}{d_2}\right)^5 = 306.7\frac{0.015}{0.026}\left(\frac{0.6}{1}\right)^5 = 13.76\,m$$

Now, by adding to the 600-mm pipe, the problem is reduced to 334.76m of 600-mm for finding the discharge through it, $\epsilon_1 = 2mm,\ H = 6\,m.$

$$6 = f\,\frac{334.76}{0.6}\frac{V^2}{2g}$$

$$f = 0.026, --\rightarrow V = 2.848 -\rightarrow\ R_e = .848 * \frac{0.6}{(2*10^{-6})} = 569600$$

From Re and $\frac{\epsilon}{d} = 0.0033$ from Moody diagram *f=0.0265*

from above equation

$$V = 2.821\ \frac{m}{s}\ \rightarrow Q = \pi\,(0.3)^2\,(2.821) = 0.798\ \frac{m^3}{s}$$

7.11 Pipes in Parallel.

The second type of pipe- system is the parallel flow type, in this case as shown in Fig. 7.14 the head losses are same in any of the lines and the total flow is the sum of flow rate in each pipe.

The minor losses are added into the lengths of each pipe as equivalent lengths. From Fig. 7.14 the conditions to be satisfied are

$$\left.\begin{aligned} h_{f1} = h_{f2} = h_{f3} = \frac{p_A}{\gamma} + z_A - \left(\frac{p_B}{\gamma} + z_B\right) \\ Q = Q_1 + Q_2 + Q_3 \end{aligned}\right\} \quad (7.56)$$

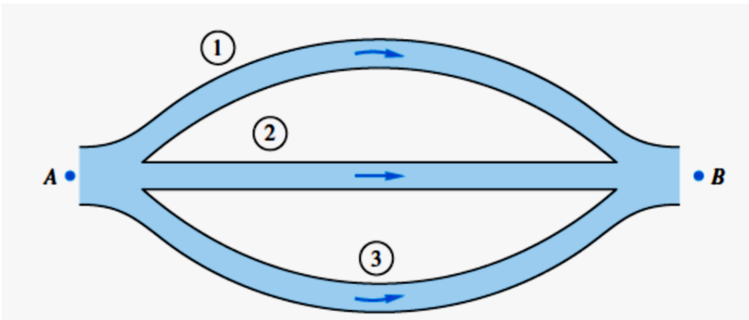


Figure 7.14: Parallel pipes system.

z_A & z_B are the elevations of point A&B.
Q is the discharge through the approach pipes.
Two types of problems occur

1) The elevations of HGL(hydraulic grid line) at A&B are known, to find the discharge.
2) Q is known, to find the distribution of flow and the head loss, size of pipe, fluid properties and roughness's are assumed to be known.

Case-1. as the simple pipe problem. Since, head loss is the drop in HGL. These discharges are added to determine the total discharge.

Case-2. The recommended procedure is as follows.

1- Assume a discharge Q', through pipe 1.
2- Solve forh'_{f1}, using assumed discharge.
3- Using h'_{f1}, find Q'_2 , Q'_3 .
4- With the three discharges for a common head loss, now assume that the given Q is split up among the pipes in the same proportion as Q'_1, Q'_2, Q'_3 ; thus

$$Q_1 = \frac{Q'_1}{\Sigma Q'} Q; \quad Q_2 = \frac{Q'_2}{\Sigma Q'} Q; \quad Q_3 = \frac{Q'_3}{\Sigma Q'} Q \quad (7.57)$$

5- Check the correctness of these discharges by computing h_{f1}, h_{f2}, h_{f3} for the computing Q_1, Q_2, Q_3 .

Ex.11

In Fig. 7.14, $L_1 = 3000\ ft\ , d_1 = 1\ ft\ , \in_1 = 0.001\ ft$
$L_2 = 2000\ ft\ , d_2 = 8in\ , \in_2 = 0.0001\ ft$
$L_3 = 4000\ ft\ , d_3 = 16\ in\ , \in_3 = 0.0008\ ft$
$\rho = 2.00 \frac{Slugs}{ft^3},\ \boldsymbol{\nu} = 0.00003 \frac{\boldsymbol{ft^2}}{\boldsymbol{s}}$
$p_A = 80\ psi, z_A = 100\ ft\ , z_B = 80\ ft$.
For a total flow of 12 cfs, determine the flow through each pipe and the pressure at B.

Sol.

For pipe-1-

Assume $Q_1' = 3\ cfs\ ; then\ V_1' = 3.82 \frac{ft}{s}$

$$\left.\begin{aligned} \therefore\ Re_1' &= \frac{3.82*1}{0.00003} = 127000 \\ \frac{\in_1}{d_1} &= 0.001 \end{aligned}\right\} \quad \text{From Moody chart} \quad f_1' = 0.022$$

$\therefore h_{f1}' = 0.022 \frac{3000}{1.0} \frac{3.82^2}{64.4} = 14.97\ ft$

For pipe-2-

$14.97 = f_2' \frac{2000}{0.667} \frac{V_2'^2}{2g} ------ (a)$

Assume $f_2' = 0.020$ (Recomended fully turbulent flow)

$then\ V_2' = 4.01 \frac{ft}{s} --\rightarrow\ Re_2' = \frac{4.01*\frac{2}{3}*1}{0.00003} = 89000$

$\frac{\in_2}{d_2} = 0.00015$

From Moody chart → $f_2' = 0.019$

Then from Eq. (a) $V_2' = 4.11 \frac{ft}{s} --\rightarrow\ Q_2' = 1.44\ cfs$

For pipe -3-

$14.97 = f_3' \frac{4000}{1.333} \frac{V_3'^2}{2g} ------ (b)$

Assume $f_3' = 0.019\ then\ V_3' = 4.01 \frac{ft}{s} --\rightarrow\ Re_3' = \frac{4.01*1.333}{0.00003} = 178000$

$\frac{\in_3}{d_3} = 0.0006$

From Moody chart. At $\left(Re_3' \& \frac{\in_3}{d_3}\right) --\rightarrow\ f_3' = 0.02$ from Eq.(b). $V_3' = 4.0 \frac{ft}{s} \&\ Q_3' = 5.6\ cfs$

The total discharge for the assumed conditions is

$\sum Q' = 3.0 + 1.44 + 5.6 = 10.04\ cfs$

From Eq. 7.57

$Q_1 = \frac{Q_1'}{\sum Q'} . Q = \frac{3.00}{10.04}\ 12 = 3.58\ cfs$
$Q_2 = \frac{1.44}{10.04}\ 12 = 1.72\ cfs$
$Q_3 = \frac{5.6}{10.04}\ 12 = 6.7\ cfs$
Check the values of h_{f1}, h_{f2}, h_{f3}

$$V_1 = \frac{3.58}{\pi * \frac{1}{4}} = 4.56\frac{ft}{s} - -\rightarrow\ Re_1, \frac{\in_1}{d_1} = 152000\} - \rightarrow\ f_1 = 0.021$$

h_{f1}=20.4ft

$$V_2 = \frac{1.72}{\frac{\pi}{9}} = 4.93\frac{ft}{s} - -\rightarrow \frac{\in_2}{d_2}\ Re_2 = 109200\} - \rightarrow\ f_2 = 0.019$$

h_{f2}=21.6 ft

$V_3 = \frac{6.7}{\frac{4\pi}{9}} = 4.8\frac{ft}{s} - \rightarrow \frac{\in_3}{d_3}\ Re_3 = 213000\} - \rightarrow\ f_3 = 0.019 f_2$
h_{f3}=20.4ft

is about midway between 0.018 & 0.019. to satisfy the condition $h_{f1} = h_{f2} = h_{f3}$
$\therefore if\ f_2 = 0.018\ then\ h_{f2} = 20.4\ ft \quad satisfying$.
To find p_B
$\frac{p_A}{\gamma} + z_A = \frac{p_B}{\gamma} + z_B + h_f$
$or \frac{p_B}{\gamma} = 80 * \frac{144}{64.4} + 100 - 80 - 20.4 = 178.1 ft$
In which the average head loss was taken. Then
$p_B = \frac{178.1*64.4}{144} = 79.6\ psi$

Problems.

P7.1 Two parallel plates kept ***100mm*** a part have laminar flow of oil between them with a maximum velocity of ***1.5 m/s*** Calculate

i) The discharge per meter width.
ii) The shear stress at the plates.
iii) The difference in pressure between two points ***20m*** a part.
iv) The velocity gradient of the plates.
v) The velocity at ***20mm*** from the plate.

Assume viscosity of oil $\mu = 24.5\ poise$.

P7.2 A liquid of viscosity oil ***0.1N.s/m²*** is filled between two parallel plates ***10mm*** a part. If the upper plate is moving at ***2m/s*** and the pressure difference between two sections ***10m*** apart is ***9.81kN/m²***. Find the relation of velocity and also determine the shear stress on the moving plate.

P7.3 What is the pipe diameter which must be used for oil flow rate ***0.0222m³/s*** at ***15.6 C°*** the energy loss per unit weight $h_f = 22.0m$ for a horizontal pipe with length ***1000m***. Using ***υ=0.00021m²/s*** & relative density= ***0.912kg/m³***.

P7.4

a) Calculate the shear stress at the wall of pipe if pipe diameter is ***305mm*** and water head loss is ***15m*** per weight of ***300m*** length.
b) Calculate the shear stress at a point ***51mm*** from centerline axis of pipe.
c) Determine the average velocity when ***f=0.05.***

P7.5 The dynamic viscosity of oil is ***0.1Pa.s*** and relative density is ***0.85***. The oil flow rate is $(44.4 * 10^{-3})$***m³/s*** through a pipe with ***305mm*** diameter and ***3048m*** length. Determine the energy loss per unit weight.

P7.6 A heavy oil flow from ***A*** to ***B*** through a horizontal pipe of cast–iron with ***153mm*** diameter and ***104.4m*** length. The pressure at section ***A*** is ***1.069MPa*** and at ***B*** is ***34.48kPa***. The kinematic viscosity of oil ***412.5*10⁻⁶m²/s*** and relative density ***0.918***. Determine the flow rate.

P7.7 The distance between two point ***A&B*** along pipe line is ***1224m***, pipe diameter is ***153mm***. Point ***B*** level is above point ***A*** about ***15.39m*** . The pressures at ***A&B*** are ***848kPa and 335kPa*** respectively. If the pipe manufacturing from wrought iron determines the discharge of oil flow between ***A&B***, take ***υ=3.83*10⁻⁶m²/s, ρ=854kg/m³.***

P7.8 Calculate the energy loss per unit weight in cast-iron pipe with ***278mm*** diameter and ***450m*** length when the flow are

a) Water at ***15.6C°*** and velocity is ***1750mm/s.***

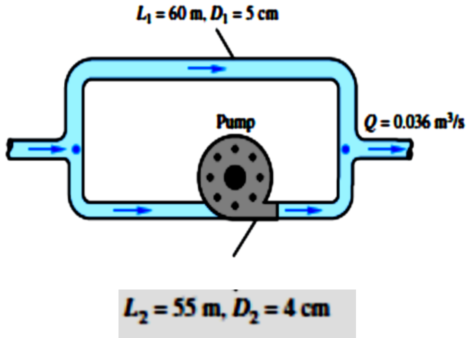
b) Oil at ***15.6C°*** and same velocity, take, $\upsilon_{water}=1.13*10^{-6} m^2/s$, $\upsilon_{oil}=4.41*10^{-6} m^2/s$.

P7.9 Determine the diameter of wrought iron pipe when the water flow rate is ***1.25m³/s*** and pipe length is ***3225m***. The drop in hydraulic grad line is ***68.8m***.

P7.10 The parallel galvanized–iron pipe system as in figure delivers water at ***20C°*** with total flow rate of ***0.036m³/s***. If the pump is wide open and not running, with a loss coefficient ***k= 1.5***, determine

a) The flow rate in each pipe.

b) The overall pressure drop.



P7.11 Show that the discharge per unit width between two parallel plate distance (***t***) a part. When one plate is moving at velocity ***U*** while the other one is held stationary for the condition of zero shear stress at the fixed plate is ***q=Ut/3***.

P7.12 An oil of viscosity ***9poise*** and ***S.G.***= ***0.9*** it is flowing through a horizontal pipe of ***60mm*** diameter. If the pressure drop in ***100m*** length of the pipe is ***1800kN/m²*** determine.

i. The rate of flow oil.

ii. The centerline velocity.

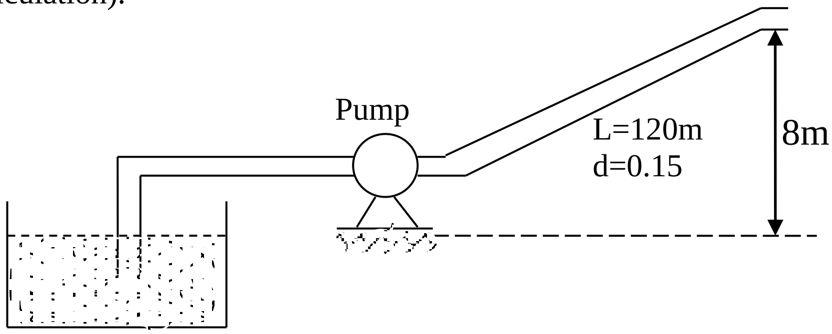
iii. Total frictional drag over ***100m*** length.

iv. The power required to maintain the flow.

v. The velocity gradient at the pipe wall.

vi. The velocity and shear stress at ***8mm*** from the wall.

P7.13 Oil flow in the pipe as shown in the figure has ***S.G.= 1.26*** and viscosity ***1.5N.s/m²*** and consider the head of pump is ***40m***. Find the flow rate in the pipe, Hint (the head loss after the pump is enter in calculation).



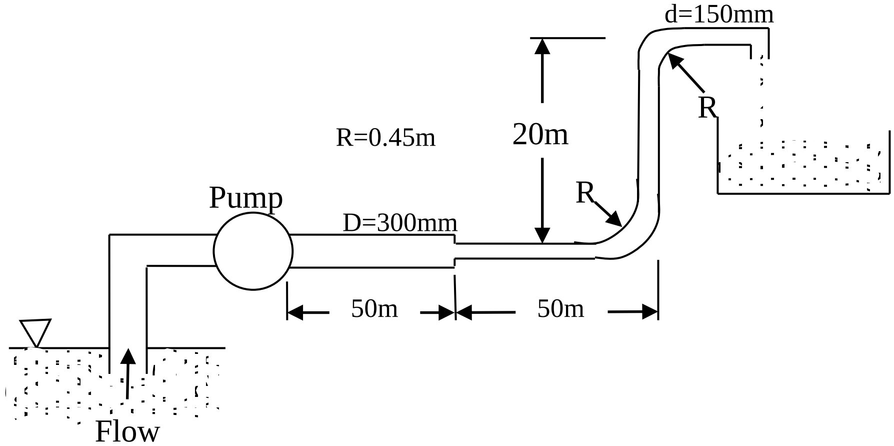
P7.14 Find the head loss of flow through a pipe the flow rate in the pipe is ***200L/s***, and the pipe made from stainless steel. The fluid has ***S.G. = 0.9*** and dynamic viscosity ***0.07 N.s/m²***. The diameter of pipe is ***300mm*** and length of pipe is ***200m***.

P7.15 Oil flow through a pipe has ***300mm*** diameter commercial steel pipe. If the head loss for ***400m*** pipe length is about ***10m***. Determine the flow rate in the pipe. Consider ***S.G.=0.9, μ=0.09 N.s/m²***.

P7.16 The system in the following figure consists a tank volume ***20m³*** and is filled after ***1000s***. If the friction in pipes is 0.03 and $\in = 0.000075\ m$ find,

i) The pressure developed by the pump on its delivery side.

ii) The power delivered to water by the pump.



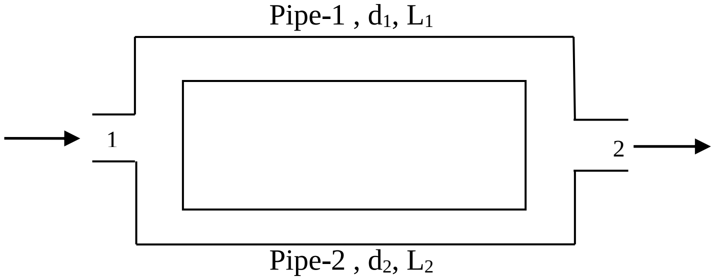
P7.17 Three pipes of diameters ***300mm***, ***200mm*** and ***400mm*** and lengths ***450m***, ***255m*** and ***315m*** respectively are connected in series between two tanks. The difference in water surface levels in two tanks is ***18m***. Determine the rate of flow of water if the coefficients of friction for three pipes are ***0.03***, ***0.0312*** and ***0.0288*** respectively considering

i) Minor losses.
ii) Neglecting minor losses.

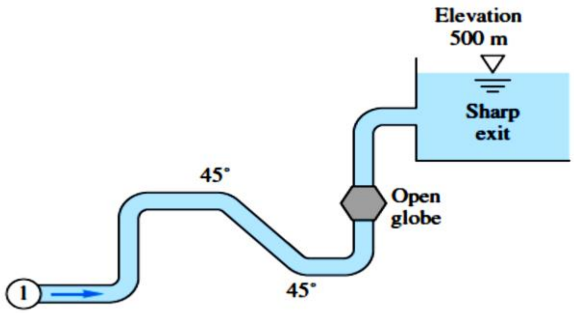
P7.18 Solve the above problem by the equivalent pipe technique consider the diameter of equivalent pipe is ***0.2m*** and ***f=0.0312***. Assume with minor losses.

P7.19 Two pipes have diameters ***200mm***, ***250mm*** and lengths ***200m*** and ***150m*** respectively connected in parallel as shown in figure. If the total flow rate is $0.1 m^3/s$ and the points (1) and (2) have same elevation and same diameters calculate

i) the flow rate in each pipe.
ii) pressure drop between (1) of (2) assume the fluid is oil has ***S.G.=0.9*** and dynamic viscosity = ***0.002N.s/m²*** and $\in$ for two pipe is ***0.25mm***



P7.20 the following figure consists of ***1250m*** of ***5.8cm*** cast-iron pipe, two ***45°*** regular screwed type and four ***90°*** screwed long-radius elbows, a fully open screwed globe valve and a sharp exit into reservoir. If the elevation at point ***1*** is ***425m***, what gage pressure is required at point ***1*** to deliver ***0.0045m³/s*** of water at ***25°C*** into reservoir?



CHAPTER 8

Introduction to Boundary Layer

8.1 Boundary Layer Definitions and Characteristics.

Boundary Layer is a small region developing around a boundary surface of body, in which the velocity of the flowing fluid increase rapidly from zero at the boundary surface and approaches the velocity of main stream. The layer adjacent to the boundary surface is known as boundary Layer (B.L.). Firstly Introduced by L.Prandtl in 1904. Fluid medium around bodies moving in fluids, can be divided into following two regions

(i) A thin layer adjoining the boundary called B.L where the viscous shear takes place.

(ii) A region outside the boundary layer where the flow behavior is quite like that of an ideal fluid and the potential flow theory is applicable.

U_∞ : is the velocity at the outer edge of the B.L.

δ: is called the dynamic B.L thickness where u=0.99U_∞ as shown in Fig. 8.1.

T_w : is the wall temperature where the fluid immediately at the surface is equal to the temperature of the surface.

δ_T: is the thermal B.L thickness, where the temperatures are changing as $T = T_w\ at\ y = 0\ T = T_\infty\ at\ y = \delta_T$

In dynamic B.L. u=u(y) u=0 at y=0; U=U_∞ $at\ y = \delta,$

$\delta = \delta(x)\quad and\ \ \delta_T = \delta_T(x)$

$\tau_0 = \mu(\frac{\partial u}{\partial y})_w$ is the shear stress at the wall. (8.1)

The displacement of the streamlines (δ_d) in the free stream as a result of velocity deficits in the B.L is known the displacement thickness. The momentum layer thickness (θ) is the equivalent thickness of a fluid layer with velocity U_∞ with momentum equal to the momentum lost due to friction , and is defind as the momentum thickness θ.

$q_w = -k\left(\frac{\partial T}{\partial y}\right)_w$ is the heat flux at the wall. (8.2)

Both (τ_w, q_w) are function of distance from the leading edge

$\tau_w = \tau_w(x), q_w = q_w(x)$

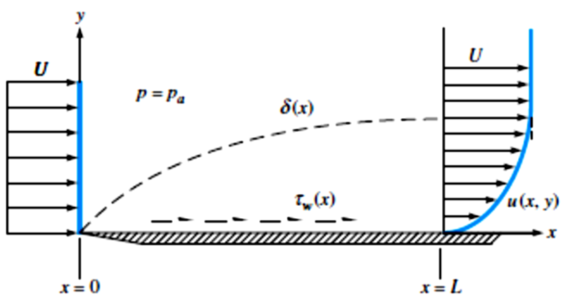


Figure 8.1: Growth of a boundary layer on a flat plate.

8.2 Boundary Layer Theory (Flow over Flat Plate).

For flow over a flat plate with zero pressure gradient, the transition process occurs when

$Re = \frac{U_\infty x_L}{\nu} = 3 * 10^5$. We assume that $U_\infty = 1.0\frac{m}{s}$ over a thin plate 10 long $Re_{critical} = 3 * 10^5$

$Re = \frac{U_\infty x_L}{\nu} = \frac{1.0 * x_L}{1.6 * 10^{-5}} = 3 * 10^5$

$x_L = 4.8m$. From Fig. 8.2, the following cases can be show.

(i) The thickness of B.L. (δ) increases with distance from leading edge $x, \delta --\rightarrow \delta(x)$

(ii) δ Decreases as U increases.

(iii) δ Increases as kinematic viscosity υ increases.

(iv) $\tau_0 \approx \mu\left(\frac{U}{\delta}\right); hence\ \tau_0$ decreases as x-increases. However, when B.L becomes turbulent.

v) If $Re = \frac{U_\infty x}{\nu} < 5 * 10^5$B.L is laminar (Velocity distribution is parabolic)

If $Re = \frac{U_\infty x}{V} > 5 * 10^5$ B.L is turbulent on that portion (Velocity distribution follows log law as a power law).

vi) The critical value of $\frac{Ux}{v}$ at which B.L change from laminar to turbulent depends on:

- Turbulence in ambient flow.
- Surface roughness.
- Pressure gradient.
- Plate curvature.
- Temperature difference between fluid and boundary.

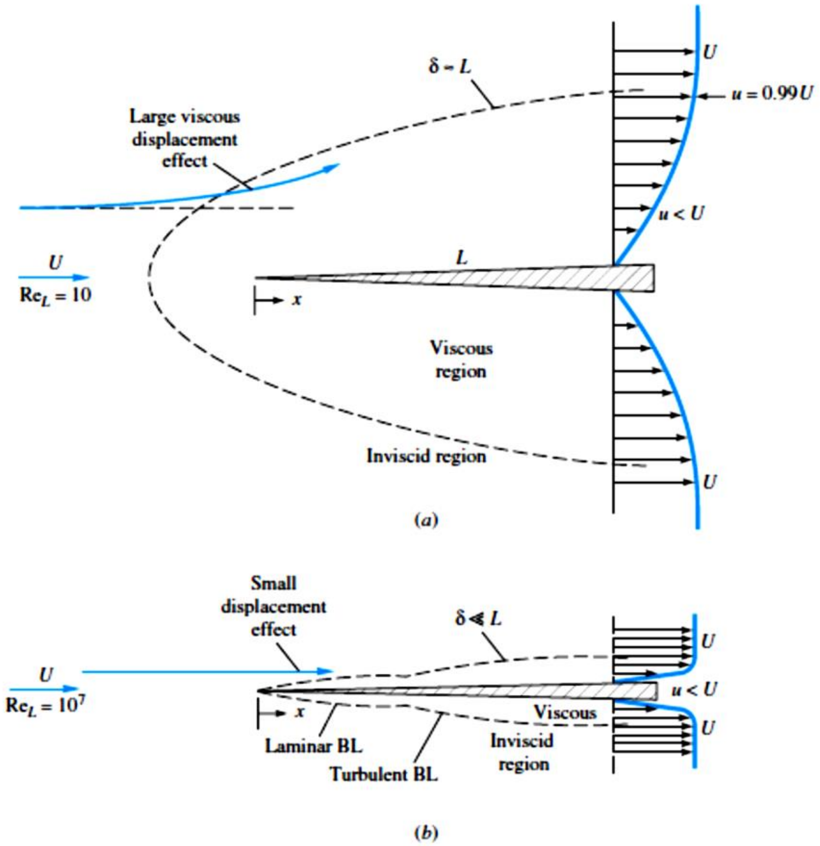


Figure 8.2: Comparison of flow past a sharp flat plate at low and high Reynolds numbers

(a) Laminar, low –Re flow; (b) high-Re flow.

8.3 Displacement Thickness (δ_d) of B.L.

Consider the mass flow per unit depth across the vertical line y=0 and $y = y_1$ as shown in Fig. 8.3.

A= actual mass flow between 0 and $y_1 = \int_0^{y_1} \rho u \, dy$

B= theoretical mass flow between 0 and y_1 if B.L were not present

B = $\int_0^{y_1} \rho_\infty U_\infty \, dy$

B-A= decrement in mass flow due to presence of B.L, i.e (missing mass flow)= $\int_0^{y_1} (\rho_\infty U_\infty - \rho u) dy - - - (a)$

Express this missing mass flow as the product of $\rho_\infty U_\infty$ and a heigh (δ_d) that is

Missing mass flow = $\rho_\infty U_\infty \delta_d - - - (b)$

Equating Eq's (a & b)

$$\rho_\infty U_\infty \delta_d = \int_0^{y_1} (\rho_\infty U_\infty - \rho u) dy$$

$$\delta_d = \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right) dy \qquad if \ \rho = \rho_\infty \ \& \ y_1 = \delta$$

$$\therefore \ \delta_d = \int_0^{\delta} \left(1 - \frac{u}{U_\infty}\right) dy \tag{8.3}$$

Physically;

1) Missing mass flow
2) Deflected the streamline upward throught a distance δ_d

$\therefore$ δ_d is the distance through which the external inviscid flow is displaced by the presence of the B.L.

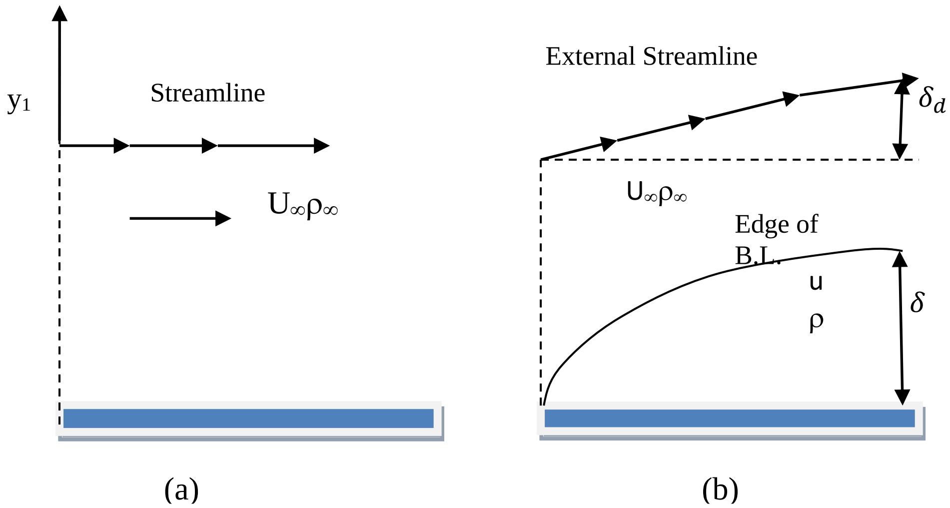


Figure 8.3: (a) Hypothetical flow with no B.L.(inviscid case).
(b)Displacement thickness in actual flow with B.L.

8.4 Momentum Thickness (θ).

To understand the physical interpretation of θ as the momentum thickness. Consider the mass flow across a segment dy as in Fig. 8.4, given by $dm = \rho u\, dy$ then

A= momentum flow across $dy = dm\, u = \rho u^2 dy - - - (a)$

If this same elemental mass flow were associated with the free- stream velocity, where the velocity is U_∞, then

B= momentum flow at free stream velocity associated with ***dm***

$= dmU_\infty = (\rho u\, dy)U_\infty - - - - - (b)$

Hence, B-A= decrement in momentum flow (missing momentum flow) associated with mass ***dm***

$= (\rho u\, U_\infty - \rho u^2)dy = \rho u\,(U_\infty - u)dy - - - (c)$

The total decrement in momentum flow across the vertical line from y=0 to $y = y_1$ is the integral of Eq. (c)

$\therefore missing\ momentum\ flow = \int_0^{y_1} \rho u(\,U_\infty - u)dy - - - -(d)$

Assume that the missing momentum flow is the product of $\rho_\infty\, U_\infty^2$ and a height θ. Then

$missing\ momentum\ flow = \rho_\infty\, U_\infty^2 \theta - - - -(e)$

Equating Eq's(d& e)

$\rho_\infty\, \mathrm{U}_\infty^2 \theta = \int_0^{y_1} \rho u(\,U_\infty - u)dy$

$\therefore\ \theta = \int_0^{y_1} \frac{\rho u}{\rho_\infty U_\infty}\left(1 - \frac{u}{\mathrm{U}_\infty}\right) dy \qquad\qquad if\ \rho = \rho_\infty\ and\ y_1 = \delta$

$$\therefore\ \theta = \int_0^{\delta} \frac{u}{\mathrm{U}_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{8.4}$$

$\therefore \theta$ is an index that is proportional to the decrement in momentum flow due to the presence of the B.L. It is the height of an ideal stream tube which is carrying the missing momentum flow at free stream conditions.

8.5 Energy Thickness($\boldsymbol{\delta_e}$).

Energy thickness is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy K.E. of the flowing fluid.

The mass of flow per second through the elementary strip= $\rho u\, dy$

K.E. of this fluid inside the B.L. $= \frac{1}{2}\, \dot{m} u^2 = \frac{1}{2}\,(\,\rho u\, dy)u^2$

K.E of the same mass of fluid before entering the B.L.

$= \frac{1}{2}\ (\rho u\ dy)\mathrm{U}_{\infty}^{2}$; Loss of K.E. through elementary strip is equal to

$= \frac{1}{2}\ (\rho u\ dy)U_{\infty}^{2} - \frac{1}{2}\ (\rho u\ dy)u^{2}$

$= \frac{1}{2}\ (\rho u)(\mathrm{U}_{\infty}^{2} - u^{2})dy$

$\therefore Total\ loss\ of\ K.E.of\ fluid = \int_0^{\delta} \frac{1}{2}\ \rho u(\ \mathrm{U}_{\infty}^{2} - u^{2})dy - - - (i)$

Let $\delta_e =$ distance by which the plate is displaced to compensate for the reduction in K.E. then loss of K.E. through $\delta_e$of fluid flowing with velocity U_∞ as follows

$= \frac{1}{2}\ (\rho\ U_{\infty}\delta_{e})\mathrm{U}_{\infty}^{2} - - - - - - (ii)$

Equating Eq's (i) and (ii), we have

$\frac{1}{2}\ (\rho U\ \delta_{e})\mathrm{U}_{\infty}^{2} = \int_0^{\delta} \frac{1}{2}\ \rho u\ (\mathrm{U}_{\infty}^{2} - u)dy$

$or\ , \delta_e = \frac{1}{\mathrm{U}_{\infty}^{3}} \int_0^{\delta} u\ (\mathrm{U}^2 - \mathrm{u}^2)\mathrm{dy}$

$$\delta_e = \int_0^{\delta} \frac{u}{\mathrm{U}_{\infty}}\left(1 - \frac{u^2}{\mathrm{U}_{\infty}^{2}}\right) dy \qquad (8.5)$$

Ex.1

The velocity distribution in the *B.L.* is given by : $\frac{u}{\mathrm{U}_\infty} = \frac{y}{\delta}$, where $\boldsymbol{u}$ is the velocity at a distance $\boldsymbol{y}$ from the plate and $u = U\ at\ y = \delta, \delta\ being\ B.L.$ thickness. Find

(i)The displacement thickness (ii) the momentum thickness (iii) the energy thickness and (iv) the value of $\delta_d\ /\theta$

Sol.

Velocity distribution $\frac{u}{\mathrm{U}_\infty} = \frac{y}{\delta}$

(i) the displacement thickness δ_d

$\delta_d = \int_0^{\delta}\left(1 - \frac{u}{\mathrm{U}\infty}\right) \mathrm{dy}$

$= \int_0^{\delta}\left(1 - \frac{y}{\delta}\right) dy$

$= \left[y - \frac{y^2}{2\delta}\right]$

$\delta_d = \left(\delta - \frac{\delta^2}{2\delta}\right) = \delta - \frac{\delta}{2} = \frac{\delta}{2}$

(ii) The momentum thickness, θ:

$$\theta = \int_0^{\delta} \frac{u}{\mathrm{U}_\infty}\left(1 - \frac{u}{\mathrm{U}_\infty}\right) dy$$

$$= \int_0^{\delta} \frac{u}{\mathrm{U}_\infty}\left(1 - \frac{u}{\mathrm{U}_\infty}\right) dy = \int_0^{\delta}\left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy = \left[\frac{y^2}{2y} - \frac{y^3}{3\delta^2}\right]_0^{\delta}$$

$$\theta = \left[\frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2}\right] = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

(iii) The energy thickness , δ_e:

$$\delta_e = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u^2}{U_\infty^2}\right) dy$$

$$= \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y^2}{\delta^2}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^3}{\delta^3}\right) dy$$

$$\delta_e = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3} = \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4}$$

(iv) The value of $\frac{\delta_d}{\theta} = \frac{\frac{\delta}{2}}{\frac{\delta}{6}} = 3$

Ex.2

The velocity distribution in the boundary layer is given by

$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^2}{\delta^2}$ Calculate the following,

(i) The ratio of displacement thickness to B.L thickness $\left(\frac{\delta d}{\delta}\right)$,

(ii) The ratio of momentum thickness to B.L thickness $\left(\frac{\theta}{\delta}\right)$.

Sol.

i) $\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\frac{y^2}{\delta^2}\right) dy$

$$\delta_d = \left[y - \frac{3}{2}\ \frac{y^2}{2\delta} + \frac{1}{2}\ \frac{y^3}{3\delta^2}\right]_0^\delta = \left[\delta - \frac{3}{4}\frac{\delta^2}{\delta} + \frac{1}{2} * \frac{\delta^3}{3\delta^2}\right]$$

$$\delta_d = \left(\delta - \frac{3}{4}\delta + \frac{\delta}{6}\right) = \frac{5}{12}\delta \qquad \frac{\delta_d}{\delta} = \frac{5}{12}$$

ii) $\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

$$\theta = \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^2}{\delta^2}\right)\left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\frac{y^2}{\delta^2}\right) dy$$

$$\theta = \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{9}{4}\frac{y^2}{\delta^2} + \frac{3}{4}\frac{y^3}{\delta^3} - \frac{1}{2}\frac{y^2}{\delta^2} + \frac{3}{4}\frac{y^3}{\delta^3} - \frac{1}{4}\frac{y^4}{\delta^4}\right) dy$$

$$\theta = \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \left(\frac{9}{4}\frac{y^2}{\delta^2} + \frac{1}{2}\frac{y^2}{\delta^2}\right) + \left(\frac{3}{4}\frac{y^3}{\delta^3} + \frac{3}{4}\ .\frac{y^3}{\delta^3}\right) - \frac{1}{4}\frac{y^4}{\delta^4}\right] dy$$

$$\theta = \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{11}{4}\frac{y^2}{\delta^2} + \frac{3}{2}\frac{y^3}{\delta^3} - \frac{1}{4}\frac{y^4}{\delta^4}\right] dy$$

$$\theta = \left[\frac{3}{2} * \frac{y^2}{2\delta} - \frac{11}{4} * \frac{y^3}{3\delta^2} + \frac{3}{2} * \frac{y^4}{4\delta^3} - \frac{1}{4} * \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{3}{2} * \frac{\delta^2}{2\delta} - \frac{11}{4} * \frac{\delta^3}{3\delta^2} + \frac{3}{2} * \frac{\delta^4}{4\delta^3} - \frac{1}{4} * \frac{\delta^5}{5\delta^4}\right]$$

$$= \left(\frac{3}{2}\delta - \frac{11}{12}\delta + \frac{3}{8}\ \delta - \frac{1}{20}\delta\right) = \frac{19}{120}\delta \longrightarrow \frac{\theta}{\delta} = \frac{19}{120}$$

8.6 Von Karman Integral Equation.

From the 2nd law of Newton's and momentum equation for a C.V as shown in Fig.8.4 the following equations can be written

$$\sum F = \frac{d}{dt}\int_{c.v} \rho\, \vec{V} d\forall + \int_{c.s} \rho \vec{V}(\vec{V}.n) dA.$$

$$\sum F_x = \rho\left(u + \frac{\partial u}{\partial x}dx\right)^2 dy - \rho u^2 dy + \rho\left(v + \frac{\partial v}{\partial y}dy\right)\left(u + \frac{\partial u}{\partial y}\, dy\right)dx - \rho u\, v dx$$

$$\sum F_x = \rho\left[u^2 + 2u\frac{\partial u}{\partial x}\, dx + \left(\frac{\partial u}{\partial x}\, dx\right)^2\right]dy - \rho u^2 dy + \rho v u dx +$$

$$\rho v\frac{\partial u}{\partial y}\, dy dx + \rho\, u\frac{\partial v}{\partial y} dy\, dx + \rho\frac{\partial v}{\partial y}.\frac{\partial u}{\partial y}\, (dy)^2 * dx - \rho u v\, dx$$

$$\sum F_x = \rho\left(2\, u\frac{\partial u}{\partial x} + \, u\frac{\partial v}{\partial y} + \, v\frac{\partial u}{\partial y}\right)dy\, dx$$

After disregarding second – order terms & $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ from C.E.

$$\therefore\ \sum F_x = \ \rho\left(\, u\frac{\partial u}{\partial x} + \, v\frac{\partial u}{\partial y}\right)dxdy \qquad (8.6)$$

The summation of forces on C.V may be written as

$$\sum F_x = \ p\, dy - \left(p + \frac{\partial p}{\partial x}\, dx\right)dy + \left(\tau_x + \frac{\partial \tau_x}{\partial y}\, dy\right)dx - \tau_x dx$$

$$\sum F_x = \ p\, dy - pdy - \frac{\partial p}{\partial x}dxdy + \tau_x dy + \frac{\partial \tau_x}{\partial y}dydx - \tau_x dx$$

$$\sum F_x = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y}\right)dxdy \qquad (8.7)$$

Equating Eq's (8.6 & 8.7)

$$\rho\left(u\frac{\partial u}{\partial x} + \, v\frac{\partial u}{\partial y}\right)dx\, dy = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y}\right)dxdy \qquad (8.8)$$

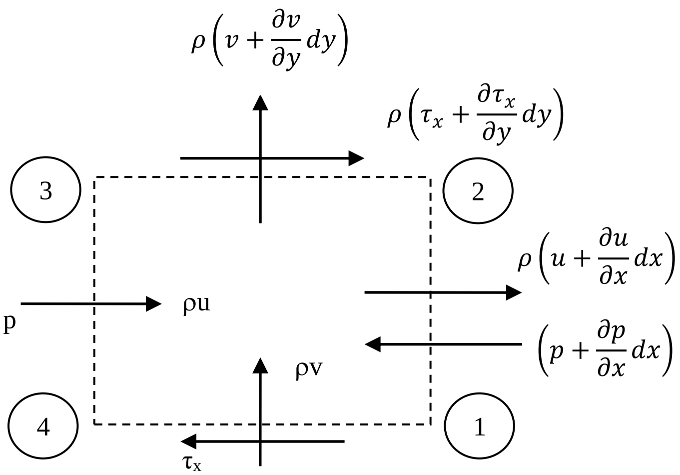


Figure 8.4: Distribution of pressure forces on control volume.

The shear stress is very nearly equal to

$\tau_x = \mu \frac{\partial u}{\partial y}$; Substituting in Eq.8.8 gives

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\frac{\mu}{\rho}\right)\frac{\partial^2 u}{\partial y^2} \qquad (8.9)$$

Table 8.1 Masses and momentum fluxes on control volume faces

Surface	Mass flux	Flux of x-momentum
(1-2)	$\rho\left(u + \frac{\partial u}{\partial x}\,dx\right)dy$	$\rho\left(u + \frac{\partial u}{\partial x}dx\right)^2\,dy$
(3-4)	$\rho u\,dy$	$\rho u^2 dy$
(2-3)	$\rho\left(v + \frac{\partial v}{\partial y}\,dy\right)dx$	$\rho\left(v + \frac{\partial v}{\partial y}\,dy\right)\left(u\frac{\partial u}{\partial y}\,dy\right)dx$
(4-1)	$\rho v\,dx$	$\rho v\,udx$

From the Integral method of momentum equation for Von Karman integral as follows

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy = \int_0^h -\frac{1}{\rho}\frac{dp}{dx}\,dy + \int_0^h v\frac{\partial^2 u}{\partial y^2}\,dy \qquad (8.10)$$

Where h is an undefined distance from the wall to outside the boundary layer. Integrating the second term in the integral by parts, we have

$$[\int_0^h v\frac{\partial u}{\partial y}dy] = [vu]_0^h - \int_0^h u\frac{\partial v}{\partial y}dy - - - - (a)$$

From C.E, v at y=h is given by ;

$$v = \int_0^h \frac{\partial v}{\partial y}dy = -\int_0^h \frac{\partial u}{\partial x}dy \qquad - - - - (b)$$

Substitute Eq. (b in a) and $u = U_\infty\ at\ y = h$

$$[\int_0^h v\,\frac{\partial u}{\partial y}\,dy] = -\mathrm{U}_\infty \int_0^h \frac{\partial u}{\partial x}\,dy + \int_0^h u\frac{\partial u}{\partial x}\,dy \qquad - - - - (c)$$

CASE (A) $\frac{\partial p}{\partial x} = 0$

Substitute Eq. (c) in Eq's (8.9& 8.10) and neglecting $\frac{\partial p}{\partial x} = 0$

$$\int_0^h \rho\left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - U_\infty\frac{\partial u}{\partial x}\right)dy = -\tau_0$$

$$\int_0^h \rho\,\left(2u\frac{\partial u}{\partial x} - \mathrm{U}_\infty\frac{\partial u}{\partial x}\right)dy = -\tau_0 \qquad - - - - - - (e)$$

$$\int_0^h \rho\,\left(\mathrm{U}_\infty\frac{\partial u}{\partial x} - 2u\frac{\partial u}{\partial x}\right)dy = \tau_0 \qquad - - - - - - (f)$$

Since $\frac{\partial u^2}{\partial x} = 2u\frac{\partial u}{\partial x}$

$\int_0^h \rho \left(\left(U_\infty \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x}(u^2) \right) dy = \tau_0$

$\tau_0 = \rho \frac{d}{dx} \int_0^h u \,(U_\infty - u) dy$ $\quad\quad$ *since* $h = \delta$

$\tau_0 = \rho \frac{d}{dx} \int_0^\delta u \,(U_\infty - u) dy$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx} \int_0^\delta \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty} \right) dy$ (8.11)

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d\theta}{dx}$ (8.12)

Since $\theta = \int_0^\delta \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty} \right) dy$ (8.13)

Eq. 8.13 is the Von Karman equation without pressure gradient dp/dx=0

CASE (B) $\frac{\partial p}{\partial x} \neq 0$

When the pressure gradient in Eq's (8.9 or 8.10) is included and from adding it in Eq. (e) as followes

$-\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$ *from Bournalli's Equation*

Now $\int_0^h \rho \left(- U_\infty \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial x} - U_\infty \frac{dU_\infty}{dx} \right) dy = -\tau_0$ (8.14)

$\frac{\partial}{\partial x}(u\, U_\infty) = U_\infty \frac{\partial u}{\partial x} + u \frac{\partial U_\infty}{\partial x}$

$U_\infty \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(u\, U_\infty) - u \frac{\partial U_\infty}{\partial x}$ Substituted in Eq. 8.14 after multiply by (-1)

$\int_0^h \left[2u \frac{\partial u}{\partial x} - U_\infty \frac{\partial U_\infty}{\partial x} + u \frac{\partial U_\infty}{\partial x} - \frac{\partial}{\partial x}(u\, U_\infty) \right] dy = -\frac{\tau_0}{\rho}$

In above eqn. $2u \frac{\partial u}{\partial x} = \frac{\partial u^2}{\partial x}$

$\int_0^h \frac{\partial}{\partial x} [\, u\,(u - U_\infty \,] dy - \frac{dU_\infty}{dx} \int_0^h (U_\infty - u \,) dy = -\frac{\tau_0}{\rho}$

$\theta = \int_0^\delta \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty} \right) dy$

$\delta_d = \int_0^\delta \left(1 - \frac{u}{U_\infty} \right) dy$

$\therefore \frac{\tau_0}{\rho} = \frac{d}{dx}(U_\infty^2 \theta) + U_\infty \delta_d \frac{dU_\infty}{dx}$

$\frac{\tau_0}{\rho} = U_\infty^2 \frac{d\theta}{dx} + \theta * 2U_\infty \frac{dU_\infty}{dx} + U_\infty \delta_d \frac{dU_\infty}{dx}$

$\frac{\tau_0}{\rho} = U_\infty^2 \frac{d\theta}{dx} + (2\theta + \delta_d) U_\infty \frac{dU_\infty}{dx}$ (8.15)

We assume

$C_f = \frac{\tau_0}{\frac{1}{2}\rho\, U_\infty^2}$ *and* $H = \frac{\delta_d}{\theta}$

C_f is the friction factor and H is the shape factor.

$$\therefore \frac{C_f}{2} = \frac{d\theta}{dx} + \theta(2+H)\ \frac{1}{U_\infty}\frac{dU_\infty}{dx} \qquad (8.16)$$

From Eq's (8.15 and 8.16) the case of $\frac{dp}{dx} = 0$ & $\frac{dU_\infty}{dx} = 0$

No gradient of pressure along the x-axis

$$\frac{\tau_0}{\rho U_\infty^2} = \frac{d\theta}{dx} = \frac{1}{2}\ cf \qquad (8.17)$$

8.7 Approximate Solution to the Laminar B.L.

We have four conditions that proposed velocity profile should satisfy on flat plat with zero pressure gradients.

$u = 0\ at\ y = 0$

$u = U_\infty\ at\ y = \delta$

$\frac{\partial u}{\partial x} = 0\ at\ y = \delta$

$\frac{\partial^2 u}{\partial y^2} = 0\ at\ y = 0$

Let $\frac{u}{U_\infty} = A + By + Cy^2 + Dy^3$ is a cubic polynomial will satisfy the four conditions.

From above conditions

At y=0 , u=0

$\frac{u}{U_\infty} = 0 = A \qquad ------(a)$

At $y = \delta \qquad u = U_\infty$

$\frac{U}{U_\infty} = 1 = B\delta + C\delta^2 + D\delta^3 \qquad ------\ (b)$

At $y = \delta \qquad \frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial y} = B + 2cy + 3Dy^2$

$B + 2C\delta + 3D\delta^2 = 0 \qquad ------\ (c)$

At y=0

$\frac{\partial^2 y}{\partial y^2} = \ 0 = 2C + 6Dy ---\rightarrow C = 0 \qquad ----(d)$

From Eq. (c) → $B = -3D\delta^2 \qquad ------\ (e)$

From Eq. (b) $B = \frac{1}{\delta} - D\delta^2 \qquad ----(f)$

Equating (e & f)

$-3D\delta^2 = \frac{1}{\delta} - D\delta^2 \qquad ----\rightarrow D = -\frac{1}{2\delta^3}\ ; \quad B = \frac{3}{2\delta}$

Hence a good approximation for the velocity profile in a laminar flow is

$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

Let us now use this velocity profile to find $\delta(x) and\ \tau_0(x)$. From Von Karman's integral Eq's (8.11 & 8.12)

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$

Assume the velocity take the following form $\frac{u}{U_\infty} = \frac{3y}{2\delta} - \frac{y^3}{2\delta^3}$ in the B.L. Calculate the thickness and the wall shear.

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{y^3}{2\delta^3}\right)\left(1 - \frac{3y}{2\delta} + \frac{y^3}{2\delta^3}\right)dy$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{9y^2}{4\delta^2} + \frac{3y^4}{4\delta^4} - \frac{y^3}{2\delta^3} + \frac{3y^4}{4\delta^4} - \frac{y^6}{4\delta^6}\right)dy$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{9y^2}{4\delta^2} + \frac{6y^4}{4\delta^4} - \frac{y^3}{2\delta^3} - \frac{y^6}{4\delta^6}\right)dy$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\left[\frac{3y^2}{4\delta} - \frac{9y^3}{12\delta^2} + \frac{6y^5}{20\delta^4} - \frac{y^4}{8\delta^3} - \frac{y^7}{28\delta^6}\right]_0^\delta$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\left[\frac{3}{4}\delta - \frac{9}{12}\delta + \frac{6}{20}\delta - \frac{\delta}{8} - \frac{\delta}{28}\right]$

$\frac{\tau_0}{\rho U_\infty^2} = \frac{d\delta}{dx}(0.1392) = 0.1392\ \frac{d\delta}{dx}$

$\therefore\ \tau_0 = 0.1392\ \rho U_\infty^2\ d\delta/dx$ (8.18)

At the wall we know that $\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$ or using the cubic profile

$\left.\frac{\partial u}{\partial y}\right|_{y=0} = B + 2Cy + 3Dy^2 = B = (\frac{3}{2\delta})U_\infty$

$\therefore\ \tau_0 = \mu\left(U_\infty \frac{3}{2\delta}\right)$ (8.19)

Equating the foregoing expression (8.18 & 8.19) for $\tau_0(x)$, we find that

$0.139\rho U_\infty^2 \frac{d\delta}{dx} = \mu\left(U_\infty \frac{3}{2\delta}\right)$

$\delta\ d\delta = \frac{\frac{3}{2}\mu}{0.139\rho U_\infty}dx = 10.8\frac{\nu}{U_\infty}dx$ (8.20)

From using at $\delta = 0\ at\ x = 0$ (the leading edge) Eq. 8.20 is integrated to give

$\delta = 4.65\sqrt{\nu x/U_\infty}\quad multiply\ by \frac{\sqrt{x}}{\sqrt{x}}$

$\delta = 4.65\frac{x}{\sqrt{Re_x}}$ (8.21)

Where Re_x is the local Reynolds number. Substituted Eq. 8.21 in Eq. 8.19 giving the wall shear as $\tau_0 = 0.323\,\rho \mathrm{U}_\infty^2 \sqrt{\frac{\nu}{x\,\mathrm{U}_\infty}}$

$$\tau_0 = \frac{0.323\rho \mathrm{U}_\infty^2}{\sqrt{Re_x}} \qquad (8.22)$$

The shearing stress is made dimensionless by dividing by $\frac{1}{2}\,\rho \mathrm{U}_\infty^2$. The local skin friction coefficient C_f its

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho \mathrm{U}_\infty^2} = \frac{0.646}{\sqrt{U_\infty \frac{x}{\nu}}} = \frac{0.646}{\sqrt{Re_x}} \qquad (8.23)$$

If the wall shear is integrated over the length L, the result per unit width is the drag force.

$F_D = \int_0^L \tau_0 dx = 0.646\,\rho \mathrm{U}_\infty \sqrt{\mathrm{U}_\infty\, L\nu}$ Where τ_0 from Eq. 8.22

$$F_D = \frac{0.646\,\rho \mathrm{U}_\infty^2 L}{\sqrt{Re_L}} \qquad (8.24)$$

Or $F_D = \int_0^L \tau_0\, dx = \tau_0 . L = C_F . \frac{1}{2}\rho \mathrm{U}_\infty^2 . L$ since τ_0 from Eq. 8.23

$$\therefore\ C_F = \frac{F_D}{\frac{1}{2}\rho \mathrm{U}_\infty^2 . L} = \frac{0.646\,\rho U_\infty^2 . L}{\frac{1}{2}\rho\, \mathrm{U}_\infty^2 . L\sqrt{Re_L}} = \frac{1.292}{\sqrt{Re_L}} \qquad (8.25)$$

Where Re_L is the Reynolds number at the end of flat plate.

Ex.3

Assume a parabolic velocity profile and calculate the B.L thickness and the wall shear. Compare with those calculate above.

Sol.

The parabolic velocity profile is assumed to be

$$\frac{u}{\mathrm{U}_\infty} = A + By + Cy^2$$

With three conditions

$$u = 0\ at\ y = 0; u = \mathrm{U}_\infty\ at\ y = \delta;\ \frac{\partial u}{\partial y} = 0\ at\ y = \delta\ \ \therefore A = 0$$

$$1 = A + B\delta + C\delta^2 = B\delta + C\delta^2\ ;$$

$$0 = B + C * 2\delta$$

Then A=0; $B = \frac{2}{\delta}$; $C = -\frac{1}{\delta^2}$. The velocity profile is

$$\frac{u}{\mathrm{U}_\infty} = 2\frac{y}{\delta} - \frac{y^2}{\delta^2} - - - - - - -(a)$$

This is substituted into Von Karman's integral equation (8.11) to obtain

$$\frac{\tau_0}{\rho \mathrm{U}_\infty^2} = \frac{d}{dx}\int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left[1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right] dy$$

$$= \frac{d}{dx}\int_0^\delta \left[\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy$$

$$= \frac{d}{dx}\left[\frac{2y^2}{2\delta} - \frac{4y^3}{3\delta^2} + \frac{2y^4}{4\delta^3} - \frac{y^3}{3\delta^2} + \frac{2y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \frac{d}{dx}\left[\delta - \frac{4}{3}\delta + \frac{1}{2}\delta - \frac{1}{3}\delta + \frac{1}{2}\delta - \frac{1}{5}\delta\right]$$

$$= \frac{d}{dx}\left[1 - \frac{4}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{5}\right]\delta$$

$$= \frac{d}{dx}\left(\frac{30-25-3}{15}\right)\delta = \frac{d}{dx}\frac{2}{15}\delta$$

$$\therefore\ \tau_0 = \frac{2}{15}\rho U_\infty^2 \frac{d\delta}{dx} \qquad ------(b)$$

We also use $\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$

From Eq. (a)

$$\tau_0 = \mu\, U_\infty \frac{2}{\delta} \qquad ------(c)$$

Equating Eq's (b & c) we obtain

$$\frac{2}{15}\,\rho U_\infty^2 \frac{d\delta}{dx} = \mu U_\infty\ \frac{2}{\delta}$$

$\delta\, d\delta = 15\frac{\nu}{U_\infty}\, dx$ Using $\delta = 0\ at\ x = 0\ \ after\ integration$

$$\int_0^\delta \delta d\delta = \int_0^x 15\frac{\nu}{U_\infty}\ dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U_\infty}\ x$$

$$\therefore\ \delta = 5.48\sqrt{\frac{\nu x}{U_\infty}} \qquad ----(d)$$

This is 18% higher than the value using the cubic profile, the wall shear is found to be

$\tau_0 = \frac{2\mu U_\infty}{\delta}$ Substitute Eq. (d)

$$\tau_0 = \frac{2\mu U_\infty}{5.43}\sqrt{\frac{U_\infty}{\nu x}}$$

$$\tau_0 = 0.365\,\rho U_\infty^2\sqrt{\frac{\nu}{x\,U_\infty}} = \frac{0.365\rho U_\infty^2}{\sqrt{Re_x}}$$

This is 13% higher than the value using the cubic velocity profile.

8.8 Solution of Turbulent B.L. Power – Law Form.

The power– law form is

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/n} \qquad n = \begin{cases} 7 & Re_x < 10^7 \\ 8 & 10^7 < Re_x \le 10^8 \\ 9 & 10^8 < Re_x \le 10^9 \end{cases} \qquad (8.26)$$

The Balsius formula is an empirical relation for the local friction coefficient

$$C_f = 0.046 \left(\frac{\nu}{U_\infty \delta}\right)^{\frac{1}{4}} \quad (8.27)$$

We have $C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}$ $\quad --\rightarrow \; \tau_0 = C_f \frac{1}{2}\rho U_\infty^2$

$$\therefore \; \tau_0 = 0.023\, \rho U_\infty^2 \left(\frac{\nu}{U_\infty \delta}\right)^{1/4} \quad (8.28)$$

Substitute the velocity profile in Von Karman's integral Eq. 8.11 with $Re_x < 16^7$

$$\frac{\tau_0}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \left[\left(\frac{y}{\delta}\right)^{1/7}\left(1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right)\right] dy$$

$$\tau_0 = \frac{7}{72}\, \rho U_\infty^2 \frac{d\delta}{dx} \quad (8.29)$$

Equating Eq's (8.28 & 8.29) for τ_0, we find that

$$\delta^{\frac{1}{4}}\, d\delta = 0.237 \left(\frac{\nu}{U_\infty}\right)^{\frac{1}{4}} dx$$

Assuming a turbulent flow from the leading edge L>>x_L , from integration the above eqn.

$$\frac{\delta^{\frac{5}{4}}}{\frac{5}{4}} = 0.237 \left(\frac{\nu}{U_\infty}\right)^{1/4} x$$

After taking the root (4/5) and multiplied by $(x/x)^{1/5}$ gives the following

$$\delta = 0.38\, x\, (Re_x)^{-1/5} \quad (8.30)$$

Substituting Eq. 8.30 for δ in to Eq. 8.27 we find that

$$C_f = 0.059(R_e)^{-1/5} \quad for \quad Re \; < 10^7 \quad (8.31)$$

The drag force $= F_D = \tau_0\, A = C_f \frac{1}{2}\, \rho U_\infty^2 (LW)$ (8.32)

Where W is the width of plate and C_f from equation 8.31.

Ex.4

Estimate the boundary layer thickness at the end of a 4-m-long flat surface if the free-stream velocity $U_\infty = 5\frac{m}{s}$. Use atmospheric air at $30C^\circ$. And predict the drag force if the surface is 5m wide

a) Neglect the laminar portion of the flow

b) Account for the Laminar portion using $Re_{crit.} = 5 * 10^5$.

Sol.

a) Let us first assume turbulent flow from the leading edge. The B.L thickness is given by Eq. 8.30. It is

$\delta = 0.38\, x\, {Re_x}^{-1/5}$

$\delta = 0.38 * 4 * (\frac{5*4}{1.6*10^{-5}})^{-1/5} = 0.092m$

The drag force is using Eq's (8.31 & 8.32); $Re \quad = \frac{U_\infty . L}{\nu}$

$F_D = C_f * \frac{1}{2}\rho U_\infty^2 (LW) = 0.059(Re \quad)^{-1/5} * \frac{1}{2}\rho U_\infty^2 (L.W)$

$F_D = 0.059\left(\frac{5*4}{1.6*10^{-5}}\right)^{-1/5} * \frac{1}{2} * 1.16 * 5^2 * 4 * 5 = 1.032N$

$Re_L = \frac{5*4}{1.6*10^{-5}} = 1.25 * 10^6$. Hence, the calculation is acceptable

The distance is found as follows

$Re_{crit.} = 5 * 10^5 = \frac{U_\infty x_L}{\nu}$

$\therefore\ x_L = \frac{5*10^5*1.6*10^{-5}}{5} = 1.6\, m$

The B.L thickness at x_L is found from Eq. 8.21 with

$\delta = 4.65\frac{x_L}{\sqrt{Re_x}} = \frac{4.65*1.6}{\sqrt{5*10^5}} = 0.0105\, m$

To find the origin of turbulent flow, using equation of B.L thickness in turbulent as

$\delta = 0.38\, x\, (Re_x)^{-1/5} = \frac{0.38x\prime}{\left(\frac{U_\infty x}{\nu}\right)^{\frac{1}{5}}} = \frac{0.38x^{\frac{4}{5}}}{\left(\frac{U_\infty}{\nu}\right)^{\frac{1}{5}}}$

$\therefore\ x'^{4/5} = \frac{\delta}{0.38}\ (\frac{U_\infty}{V})^{1/5}$ where $\delta =$ the same at end of L.B

$x' = \left(\frac{0.0105}{0.38}\right)^{5/4}\left(\frac{5}{1.6*10^{-5}}\right)^{1/4} = 0.2663m$

The distance x_t as in figure is then

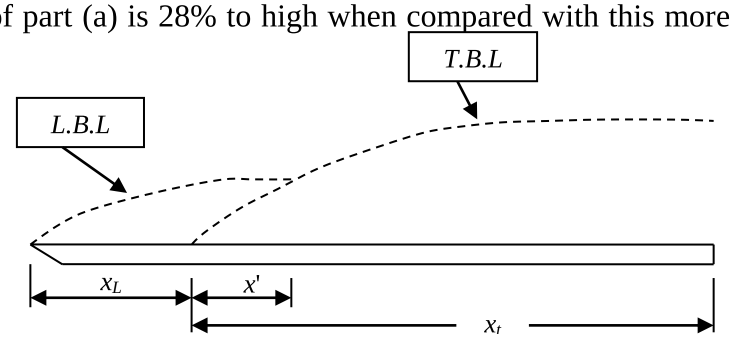
$x_t = L - x_L + x' = 4 - 1.6 + 0.266 = 2.666M$

To find the B.L thickness at the end of plate using Eq. 8.30

$\delta = 0.38\, x\, Re^{-1/5} = 0.38 * 2.666 * \left(\frac{5*2.666}{1.6*10^{-5}}\right)^{-1/5}$

$\delta = 0.0662m$

The value of part (a) is 28% to high when compared with this more accurate value.



Problems.

P8.1 How far from the leading edge can turbulence be expected on an airfoil traveling at 100 m/s if the kinematic viscosity at different elevation as followes,

$\nu_1 = 1.169 * 10^{-5}\ m/s^2 at\ T = -20C^{\circ}$

$\nu_2 = 1.087 * 10^{-5}\ m/s^2\ at\ T = -30^{\circ}$

$\nu_3 = 1.008 * 10^{-5}\ m/s^2\ \ at\ T = -40^{\circ}$ use $Re_{crit} = 6 * 10^5$ and assume flat plate with zero pressure gradient.

[x_{L1}=0.1014 m, x_{L2}=0.065 m, x_{L3}=0.06]

P8.2 Assume that $u = \mathrm{U}_\infty \sin\left(\frac{\pi y}{2\delta}\right)$ in a zero pressure gradient boundary Layer. Calculate; $(a)\ \ \delta(x),\ \ \ (b)\ \ \ \tau_0(x)$.

P8.3 A boundary Layer profile is approximated with

$u = 3U_\infty \frac{y}{0}$ $\qquad 0 < y \le \delta/6$

$u = U_\infty\left(\frac{y}{\delta} + \frac{1}{3}\right)$ $\qquad \frac{\delta}{6} < y \le \frac{\delta}{2}$

$u = U_\infty\left(\frac{y}{3\delta} + \frac{2}{3}\right)$ $\qquad \frac{\delta}{2} < y \le \delta$

Determine $\delta(x) and\ \tau_0(x)$.

P8.4 A Laminar flow is maintained in a B.L on a 6m-long 5m-wide flat plate with 20C° atmospheric air flowing at 4 m/s. Assuming a parabolic profile. Calculate

(a) δ at *x*=6m. ***[5.84 mm]***

(b) $\tau_0\ at\ x = 6m$. ***[0.025 N/m^2]***

(c) The drag force on one side. ***[0.75 N]***

P8.5 The velocity profile at a given x-location in B.L is assumed to be $u(y) = 10\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$. A stream is 2 cm from the flat plate at the leading edge. How far is it from the plate when *x=3m* (i.e what is h)? Also, calculate the displacement thickness at *x =3m*. Compare the displacement thickness to (h=2 cm).

$[\mathbf{2.89}\ \boldsymbol{cm}, \boldsymbol{\delta_d} = \mathbf{0.894}\ \boldsymbol{mm}]$

P8.6 Atmospheric air at 20C° flows at 10 m/s over a 2m-long 4m-wide flat plat. Calculate the maximum B.L thickness and drag force on one side assuming

a) Laminar flow over the entire length. $[\boldsymbol{\delta = 2.14\ mm, F_d = 1.38\ N}]$
b) Turbulent flow over the entire length. $[\boldsymbol{\delta = 45\ mm, F_d = 1.687\ N}]$

P8.7 Fluid flows over a flat plat at 20 m/s. Neglecting the laminar portion then determine δ and τ_0 $at\ x = 6\ m\ if\ the\ fluid\ is$:

a- Atmospheric air at 20C° ***[δ=95 mm, τ_0=17.64 N/m^2]***

b- Water at 20C° ***[δ=59 mm, τ_0=279.44 N/m^2]***

P8.8 Find the shear stress and the thickness of the boundary layer

i- at the trailing edge of smooth flat plate 3.0 m wide and 0.6 m long parallel to flow immersed in 15 C° water flowing at the an undisturbed velocity of 0.9 m/s. Assume a laminar boundary layer over the whole plate. ***[δ=4 mm, τ_0=0.38 N/m^2]***

ii- At the center. ***[δ=2.86 mm, τ_0=0.537N/m^2]***

iii- Find the total friction drag on one side of the plate. ***[F_d=0.684 N]***

P8.9 i- Given the general equation for a parabola u=ay^2+by+c, drive the dimensional velocity distribution equation (u/U).

ii- Determine the shear stress at 150 mm and 300 mm back from the leading edge of the plate placed longitudinally in oil (S.G=0.925, $\upsilon=0.73*10^{-4} m^2/s$) flowing with undisturbed U=0.6 m/s.

[τ_0=3.052 N/m^2, τ_0=2.16 N/m^2]

P8.10 A 80 m long streamlined train has 2.3 m high sides and 2.4 m wide top. Determine the power required to overcome the skin – friction drag when the train is traveling at 15 m/s through standard atmosphere at sea level, assuming the drag on the sides and top to be equal to that on one side of a flat plate 5m wide and 80 m long. ***[P=3.825 kW]***

CHAPTER 9

Turbomachinery

9.1 Introduction.

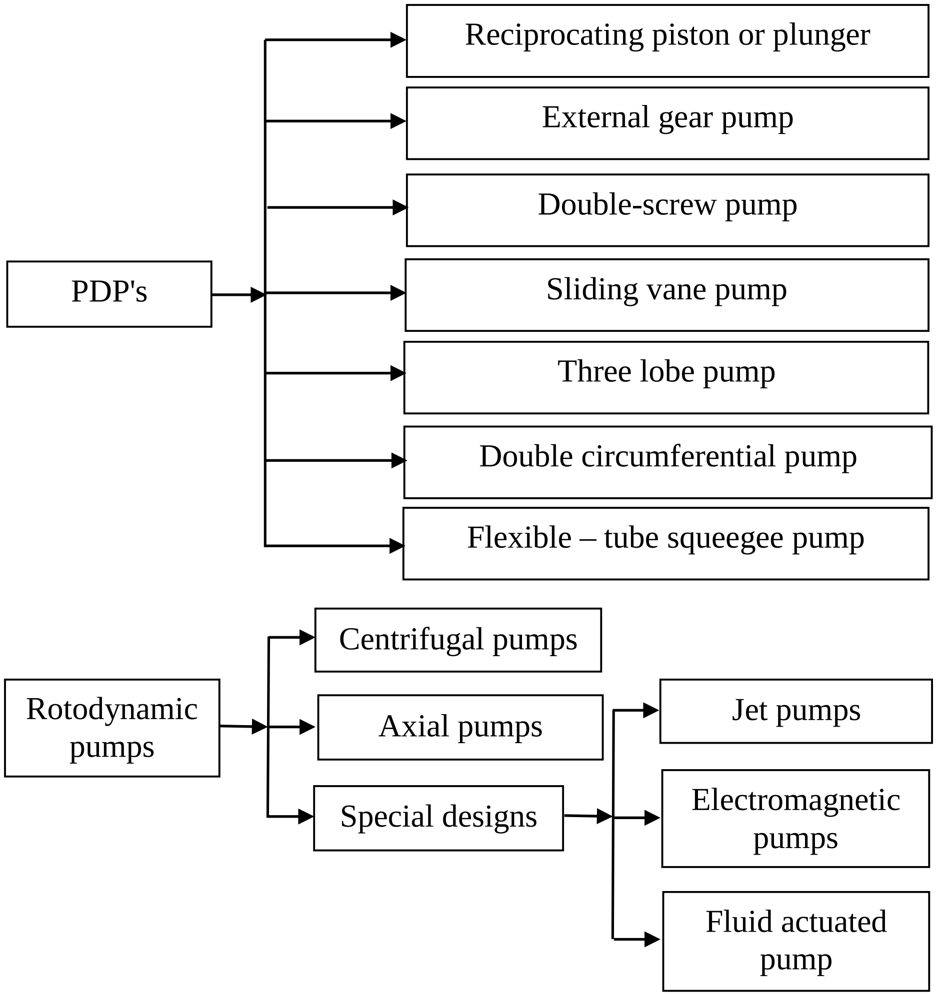
The pump family is a machine which is designed to add energy to the fluid, but the turbines family which extract energy from the fluid. Both types are usually connected to a rotating shaft, hence this is the turbomachinery.

Machine which deliver liquid are called pump, if machine delivers gases can be classified into the categories as follows

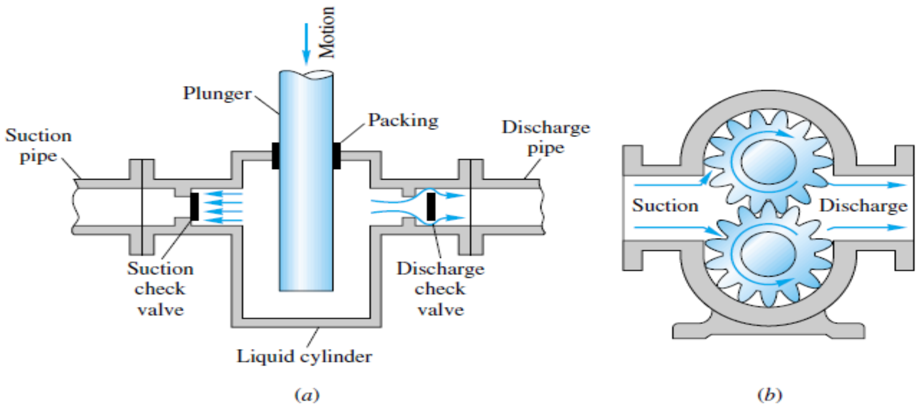
Three types of machines →
- Fan, pressure rise very small (few inches of water)
- Blower, up to 1 atm.
- Compressor above 1 atm.

If machine delivers liquid can be classified into two type as follows

pumps are used today →
- Positive–displacement pumps (PDP's) are limited by (low pressure, small volume).
- Rotodynamic or momentum change pumps. (From small volume to very large volume and higher pressures).



All types of pumps can be shows in Fig. (9.1), but this chapter is concern to the centrifugal pump type, which are explain in details in the next sections.



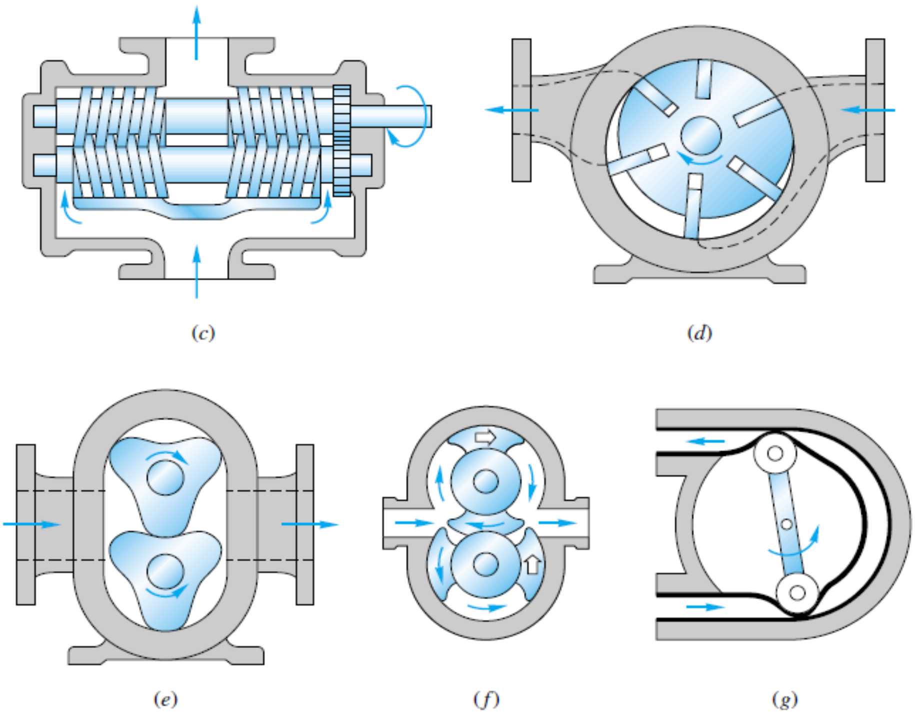


Figure (9.1) Schematic design of positive-displacement pumps: (*a*) reciprocating piston or plunger, (*b*) external gear pump, (*c*) double-screw pump, (*d*) sliding vane, (*e*) three lobe pump, (*f*) double circumferential piston, (*g*) flexible-tube squeegee.

9.2 Centrifugal Pump.

The main components of centrifugal pumps are

Centrifugal Pump →

- The impeller
- The casing
- The drive shaft with gland and packing
- Suction pipe with one way valve (foot valve)
- Delivery pipe with delivery valve

The liquid enters the eye of the impeller axially due to the suction created by the impeller motion as shown in Fig (9.2). The impeller blades guide the fluid and impart momentum to the fluid, which increase the total head (or pressure) of the fluid. The casing can be simple volute type or diffuser can be used as desired as in Fig. (9.3).

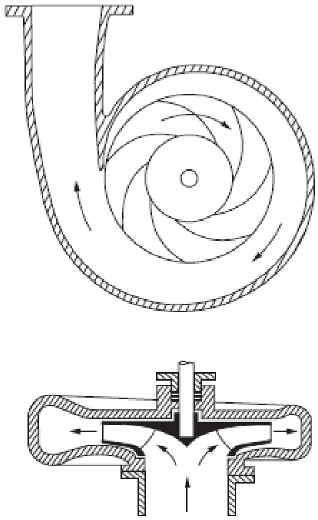


Figure 9.2 Volute type centrifugal pump.

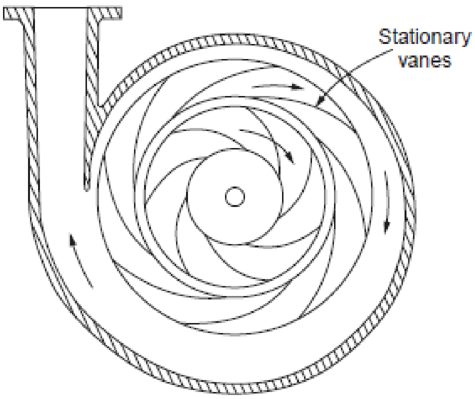


Figure 9.3 Diffuser pump.

The volute is a spiral casing of gradually increasing cross section. A part of the kinetic energy in the fluid is converted to pressure in the casing.

9.2.1 Impeller.

The impeller consists of two disc plates with blades mounted perpendicularly on its surface in between. The blades of the rotating impeller transfer energy to the fluid there by increasing pressure and velocity. The fluid enters the impeller eye through fluid sucking then flow through the impeller channels formed by the blades between hub plate and shroud plate. Fig.(9.4) shows the impeller components and the flow direction relatively to the impeller.

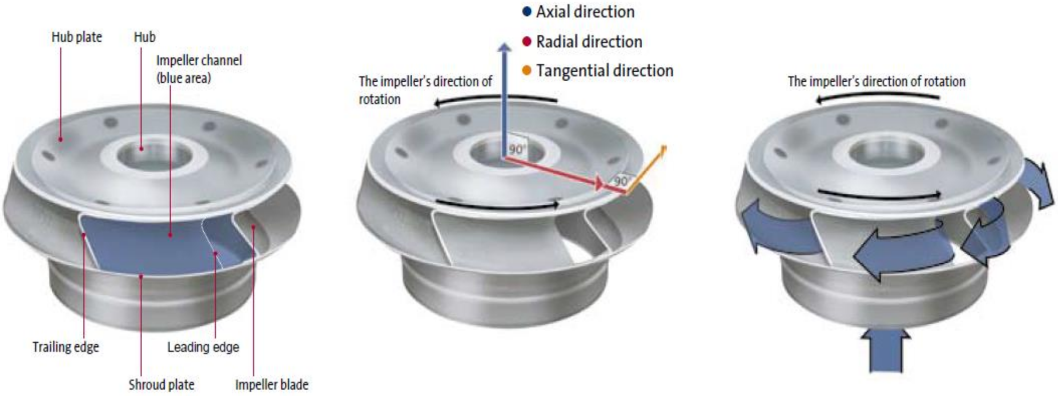


Figure (9.4) the impeller components and the flow directions.

The blades may be of three different orientations

Blade types →
- Radial type β_2=90°
- Backward type β_2<90°
- Forward type β_2>90°

Fig. (9.5) shows the blade type orientations can be used.

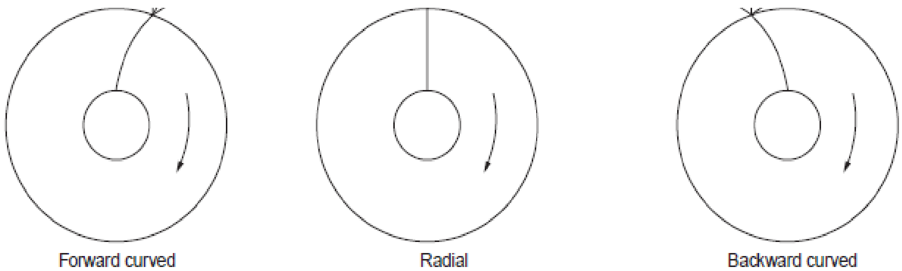


Figure (9.5) Different blade arrangements.

Also the impeller can be classified as follows,

- Impeller types
  - Open impeller type (when the blades mounted perpendicularly on disc).
  - Shrouded impeller type (when another disc is used to cover the blades).
  - Blades spreading out from the shaft.

9.2.2 Classification of centrifugal pump.

Centrifugal pump may be classified in several ways as follows,

- Basis of speed
  - Low speed
  - Medium speed
  - High speed

- Direction of flow
  - Radial flow
  - Mixed flow
  - Axial flow

- Basis of head
  - Low head (10m and below)
  - Medium head (10m – 50m)
  - High head (above 50m)

Basis of entry → Single entry
Basis of entry → Double entry

Fig. (9.6) shows the single and double entry of centrifugal pump.

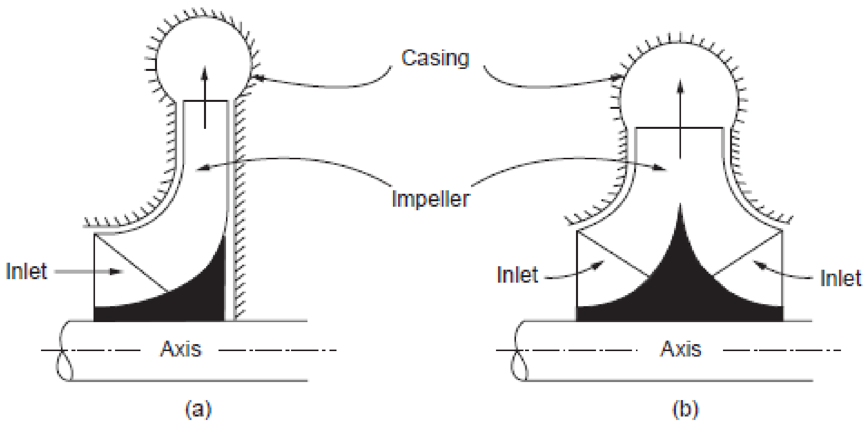


Figure (9.6) (a) Single entry, (b) Double entry pumps

9.3 Pump Head.

The main assumption in this analysis the flow is to be steady, the pump basically increases the Bernoulli head of the flow between the eye, point (1), and the exit, point (2) as show in Fig. (9.7)

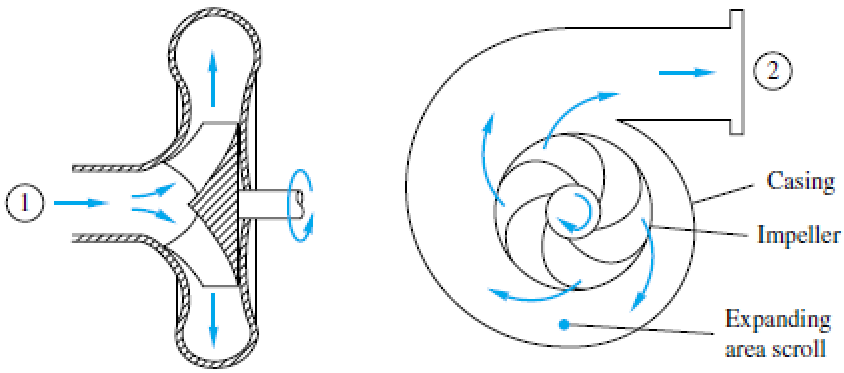


Figure (9.7) schematic of a typical centrifugal pump.

The change in head is denoted by (H).

$$H = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + Z\right)_2 - \left(\frac{p}{\rho g} + \frac{V^2}{2g} + Z\right)_1 = h_s - h_f \qquad (9.1)$$

Where h_s is the pump head supplied and h_f is the losses. Now, the primary output parameter for any turbomachine is the net head (H).

Usually $V_2 \cong V_1$ and $(Z_2 - Z_1)$ less than one meter, then, the net pump head is

$$H = \frac{p_2 - p_1}{\rho g} = \frac{\Delta p}{\rho g} \qquad (9.2)$$

The power delivered to the fluid simply equals the specific weight times the discharge times the net head change.

$$P_w = \rho g Q H \qquad (9.3)$$

This is called the water horsepower. The power required to drive the pump is the brake horsepower

$$P_{bhp} = \omega T_{sh} \qquad (9.4)$$

Where ω is the shaft angular velocity.
T_{sh} is the shaft torque.

9.4 Pump Theory.

To predict actually the head, power, efficiency and flow rate of a pump two theoretical approaches are possible

Pump Theory → Simple one-dimensional formula.
Pump Theory → Complex digital – computer models.

To construct an elementary theory of pump performance in this chapter, we assume the following → One-dimensional flow.
→ Idealized fluid – velocity vector through the impeller.
→ Angular momentum theorem for C.V.

To analyze the centrifugal pump, we choose the annular region that encloses the impeller section as the control volume C.V. as in Fig. (9.8), which shows the idealized velocity diagram and the impeller geometry.

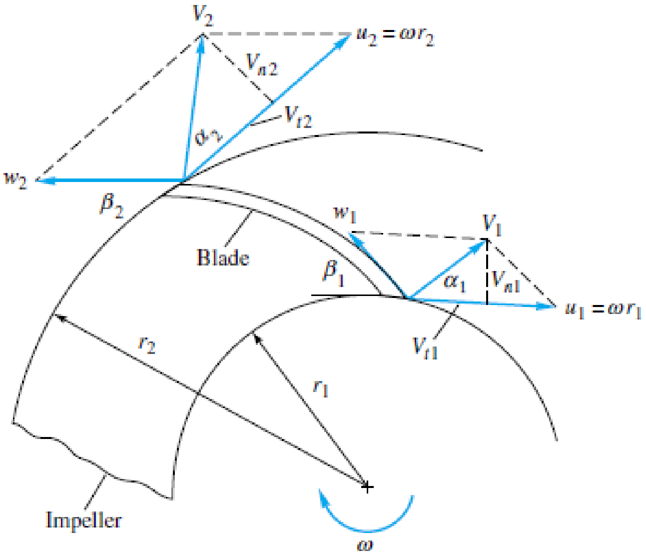


Figure (9.8) Inlet and exit velocity diagrams for an idealized pump impeller.

Since, r_1 inside radius of impeller.
r_2 outside radius of impeller.
ω The angular velocity of shaft impeller blades.

At inlet,
$u_1 = \omega r_1$ is the circumferential speed of the tip impeller at r_1.
w_1 velocity component of fluid tangent or parallel to the blade angle β_1.
V_1 is the absolute velocity of fluid at entrance, is the vector sum of w_1 and u_1.

At exit,
$u_2 = \omega r_2$ is the circumferential speed of the tip impeller at r_2.
w_2 velocity component outlet of fluid tangent or parallel to the blade angle β_2.
V_2 is the absolute velocity of fluid at exit, is the vector sum of w_2 and u_2.

The angular – momentum theorem to a radial flow devices (sec. 4.7), we arrived to a result for the applied torque T_{sh}

$$T_{sh} = \dot{m}\,(r_2 V_{t2} - r_1 V_{t1}) = \rho Q(r_2 V_{t2} - r_1 V_{t1}) \tag{9.5}$$

Where V_{t1} and V_{t2} are the absolute circumferential or tangent velocity components of the flow at inlet and exit. The power delivered to the fluid is thus,

$$P_w = \omega T_{sh} = \rho Q(u_2 V_{t2} - u_1 V_{t1}) \tag{9.6}$$

Or $$H = \frac{P_w}{\rho g Q} = \frac{1}{g}(u_2 V_{t2} - u_1 V_{t1}) \tag{9.7}$$

Eq's (9.5, 9.6 and 9.7) are the Euler turbomachine equations, showing that the torque, power and ideal head are functions only of the rotor-tip velocities u_1, u_2 and the absolute fluid tangential velocities V_{t1} and V_{t2} independent of the axial velocities.

We can rewriting these relation in other form, from the geometry of Fig. (9.6)

$V^2 = u^2 + w^2 - 2uwcos\beta \quad ; \quad w\,cos\beta = u - V_t$

Or $uV_t = \frac{1}{2}(V^2 + u^2 - w^2)$ (9.8)

Substituting Eq. (9.8) into Eq. (9.7) gives

$$H = \frac{1}{2g}[(V_2^2 - V_1^2) + (u_2^2 - u_1^2) - (w_2^2 - w_1^2)] \qquad (9.9)$$

Substituting for (*H)* from its definition in Eq. (9.1) and rearranging, we obtain the classic relation

$$\frac{p}{\rho g} + Z + \frac{w^2}{2g} - \frac{r^2\omega^2}{2g} = constant \qquad (9.10)$$

This is the Bernoulli equation in rotating coordinates and applies to either two or three-dimensional ideal incompressible flow. For a centrifugal pump, the power can be related to the radial velocity $V_n = V_t \tan\alpha$ and the continuity equation

$$P_w = \rho Q(u_2 V_{n2} cot\alpha_2 - u_1 V_{n1} cot\alpha_1) \qquad (9.11)$$

Where

$$V_{n2} = \frac{Q}{2\pi r_2 b_2} \quad \text{and} \quad V_{n1} = \frac{Q}{2\pi r_1 b_1} \qquad (9.12)$$

Where b_1 and b_2 are the blade widths at inlet and exit, with the pump parameters (r_1, r_2, β_1, β_2 and ω) are known. Eq. (9.7) or Eq. (9.11) is used to compute idealized power and head versus discharge.

The 'design' flow rate Q^* is commonly estimated by using that the flow enters exactly normal to the impeller; $\alpha_1 = 90^\circ$; $V_{n1} = V_1$

Ex.1

Given are the following data for a commercial centrifugal water pump: r_1=4 in, r_2= 7 in, β_1=30°, β_2 =20°, speed N=1440 r/min. Estimate

a- The design – point discharge.
b- The water horsepower.
c- The head if $b_1 = b_2$=1.75 in.

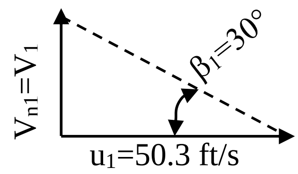
Sol.

(a) $\omega = \frac{2\pi N}{60} = \frac{2\pi(1440)}{60} = 150.8\, rad/s$

$u_1 = \omega r_1$=150.8 (4/12) = 50.3 ft/s

$u_2 = \omega r_2$=150.8 (7/12) = 88.0 ft/s

For design point $V_{n1} = V_1$ and $\alpha_1 = 90^\circ$ then the inlet velocity diagram as follows



$$\tan 30 = \frac{V_{n1}}{u_1} \quad \rightarrow \quad V_{n1} = u_1 * tan30 = 29.0\ ft/s$$

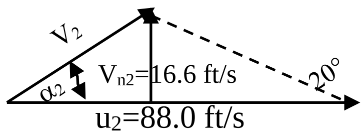
Hence the discharge is

$$Q = 2\pi r_1 b_1 V_{n1} = 2\pi\left(\frac{4}{12}\right)\left(\frac{1.75}{12}\right)(29.0) = 8.87\frac{ft^3}{s}$$

(b) The outlet radial velocity follows from Q

$$V_{n2} = \frac{Q}{2\pi r_2 b_2} = \frac{8.87}{2\pi(\frac{7}{12})(\frac{1.75}{12})} = 16.6\ ft/s$$

This enable us to construct the outlet – velocity diagram as in below



The tangential component is

$$V_{t2} = u_2 - V_{n2}\cot\beta_2 = 88.0 - 16.6\ cot20° = 42.4\ ft/s$$

$$\propto_2 = tan^{-1}\frac{16.6}{42.4} = 21.4°$$

The power is then computed as follows, with $V_{t1} = 0$ at the design point.

$$P_w = \rho Q u_2 V_{t2} = (1.94 * 8.87 * 88.0 * 42.4) = 64100\ ft\ lb/s$$

$$P_w = \frac{64100}{550} = 117\ hp$$

(c) Finally, the head is estimated by the following

$$H \approx \frac{P_w}{\rho g Q} = \frac{64100}{(62.4)(8.87)} = 116\ ft$$

9.5 Pressure Developed by the Impeller.

Fig. (9.9) shows the general system arrangement of a centrifugal pump.

Z_s Suction level above the water level.

Z_d Delivery level above the impeller outlet.

h_{fs}, h_{fd} Friction losses head in both suction and delivery sides.

V_s, V_d Fluid velocities in pipes for both sides.

Applying B.E. between the water level and pump suction.

$$\frac{p_s}{\gamma} + Z_s + \frac{V_s^2}{2g} + h_{fs} = \frac{p_a}{\gamma} \tag{9.13}$$

$$\therefore \frac{p_s}{\gamma} = \frac{p_a}{\gamma} - Z_s - \frac{V_s^2}{2g} - h_{fs} \tag{9.14}$$

Similarly applying B.E. theorem between the pump delivery and the delivery at the tank,

$$\frac{p_d}{\gamma} + \frac{V_d^2}{2g} = \frac{p_a}{\gamma} + Z_d + \frac{V_d^2}{2g} + h_{fd}$$

$$or\ \frac{p_d}{\gamma} = \frac{p_a}{\gamma} + Z_d + h_{fd} \tag{9.15}$$

Where p_d is the pressure at the pump delivery, from Eq.(9.14 &9.15)

$$\frac{p_d}{\gamma} - \frac{p_s}{\gamma} = \left(\frac{p_a}{\gamma} + Z_d + h_{fd}\right) - \left(\frac{p_a}{\gamma} - Z_s - \frac{V_s^2}{2g} - h_{fs}\right)$$

$$\frac{p_d - p_s}{\gamma} = Z_d + Z_s + \frac{V_s^2}{2g} + h_L = H_e + \frac{V_s^2}{2g} \tag{9.16}$$

Where H_e is the effective head and h_L is the total friction head.

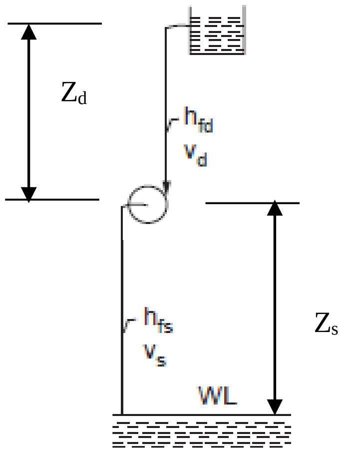


Figure (9.9) Centrifugal pump system

9.6 Manometric Head.

The official code defines the head on the pump as the difference in total energy head at the suction and delivery flanges. This head is defined as manometric head. The total energy at suction inlet or suction side expressed as suction head of fluid and termed as H_s

$$H_s = \frac{p_s}{\gamma} + \frac{V_s^2}{2g} + Z_{sd}$$

Where Z_{sd} is the height of suction gauge from datum. The total energy at the delivery side of the pump expressed as delivery head and termed as H_d

$$H_d = \frac{p_d}{\gamma} + \frac{V_d^2}{2g} + Z_{dd}$$

Where Z_{dd} is the height of delivery gauge from datum. Know, the difference in total energy of fluid is defined as H_m

$$H_m = H_d - H_s = \left(\frac{p_d}{\gamma} - \frac{p_s}{\gamma}\right) + \left(\frac{V_d^2 - V_s^2}{2g}\right) + (Z_{dd} - Z_{sd}) \qquad (9.17)$$

Substituting Eq.(9.16) in Eq. (9.17) and rearranging will be as follows

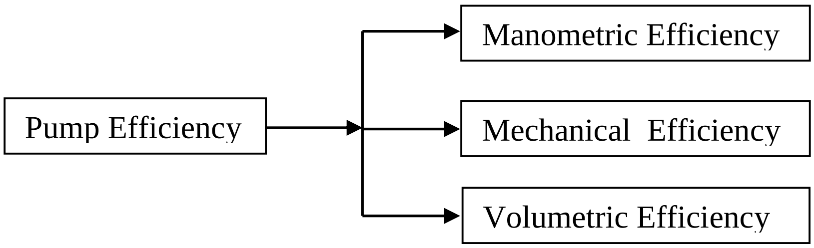
$$H_m = H_e + \frac{V_d^2}{2g} + (Z_{dd} - Z_{sd}) \qquad (9.18)$$

As $(Z_{dd} - Z_{sd})$ is small and $\frac{V_d^2}{2g}$ is also small as the gauges are fixed as close as possible.

$$\therefore\ H_m = static\ head + all\ losses \qquad (9.19)$$

9.7 Pump Efficiency.

When the losses are exists in the pump, then (***P**w*) is actually less than (***P**bhp*). The goal of the pump designer is to make the efficiency ***(η)*** as high as possible over as broad a range of discharge (Q) as possible. The efficiency can be divided to three types as follows



*A- **Manometric efficiency.***

The ideal head ***(H)*** can be expressed by Eq. (9.7)

$$H_{ideal} = \frac{1}{g}(u_2 V_{t2} - u_1 V_{t1})$$

From Fig. (9.10), the inlet tangent velocity is generally equal to zero due to no guide vanes at inlet. So the inlet triangle is right angle as shown in Fig.(9.10.a). Therefor

$V_1 = V_{n1}$ and are vertical

$$tan\beta_1 = \frac{V_1}{u_1},\ \ or\ , = \frac{V_{n1}}{u_1}$$

Since, $V_{t1} = 0.0$

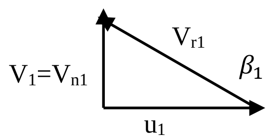
$$H_{ideal} = \frac{u_2 V_{t2}}{g} \qquad (9.20)$$

From the outlet triangle

$$u_2 = \omega r_2 = \frac{2\pi N}{60} r_2 = \frac{\pi d_2 N}{60}$$

$$tan\beta_2 = \frac{V_{n2}}{u_2 - V_{t2}}$$

$$u_2 - V_{t2} = \frac{V_{n2}}{tan\beta_2}$$



(a) Inlet Triangle

$$\therefore V_{t2} = u_2 - \frac{V_{n2}}{tan\beta_2}$$

From Eq. (9.20)

$$\therefore H_{ideal} = \frac{u_2}{g}\left(u_2 - \frac{V_{n2}}{tan\beta_2}\right)$$

Now, manometric efficiency is defined as the ratio of manometic head and ideal head

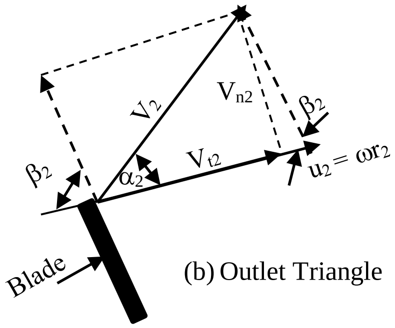


Figure (9.10) Velocity triangles for backward curved bladed pump.

$$\eta_m = \frac{H_m}{H_{ideal}} = \frac{H_m\, g}{u_2 V_{t2}} = \frac{H_m\, g}{u_2\left(u_2 - \frac{V_{n2}}{tan\beta_2}\right)} \qquad (9.21)$$

B- Mechanical Efficiency.

The mechanical efficiency is defined as

$$\eta_{mech} = \frac{Energy\ transfer\ to\ the\ fluid}{Work\ input}$$

$$\eta_{mech} = \frac{\rho Q(u_2 V_{t2})}{Power\ input} = \frac{\rho Q(u_2 V_{t2})}{P_{bhp}} \qquad (9.22)$$

C- Volumetric Efficiency.

There are always some leakage after being imparted energy by the impeller.

$$\text{Volumetric efficiency} = \frac{\text{Volum deliverd (Q)}}{\text{Volume passing throught impeller } (Q+Q_L)}$$

Where (Q_L) is the losses of fluid due to leakage, thus, the total efficiency is simply the product of its three parts.

$\eta_o = \eta_m \eta_{mech} \eta_{vol.}$ Or from basic definition can be defined as

$$\eta_o = \frac{P_w}{P_{bhp}} = \frac{\rho g Q H}{P_{bhp}} \qquad (9.23)$$

Ex.2

The following details refer to a centrifugal pump. Outer diameter 30cm, eye diameter 15 cm. blade angle at inlet (β_1=30°) blade angle at outlet (β_2=25°). Impeller speed (N=1450 rpm). The velocity remains constant. The whirl (tangent) velocity at inlet is zero. Determine (a)- the torque applied if the (η_m=0.76%). (b)- The power when the width of blades at outlet is 2 cm and equal the width at inlet. (c)- The head.

Sol.

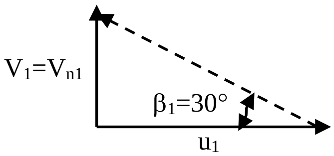
$$u_2 = \frac{2\pi r_2 N}{60} = \frac{\pi d_2 N}{60} = \frac{\pi * 0.3 * 1450}{60} = 22.78\ m/s$$

$u_1 = \frac{2\pi r_1 N}{60} = \frac{\pi d_1 N}{60} = \frac{\pi * 0.15 * 1450}{60} = 11.39\ m/s$

$V_1 = V_{n1}$ since $V_{t1} = 0$

From inlet velocity diagram

$\tan 30 = \frac{V_{n1}}{u_1} \rightarrow \ V_{n1} = u_1 * tan30 = 11.39 * tan30 = 6.58\ m/s$

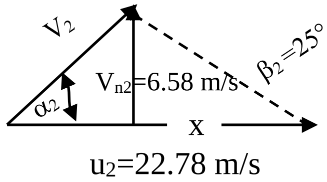


$V_{t2} = u_2 - x$

$$tan\beta_2 = \frac{V_{n2}}{X}$$

$$\therefore X = \frac{V_{n2}}{tan\beta_2}$$

$$\therefore V_{t2} = u_2 - \frac{V_{n2}}{tan\beta_2} = 22.78 - \frac{6.58}{\tan 25} = 8.67\ m/s$$



Power delivered to the fluid as

$$P_w = \omega T_{sh} = \rho Q(u_2 V_{t2} - u_1 V_{t1})$$

$$Q = 2\pi * r_2 * b * V_{n2} = \pi * d_2 * b * V_{n2} = \pi * 0.3 * 0.02 * 6.58$$
$$= 0.124\ m^3/s$$

(a).

$$T_{sh} = \dot{m}\,(r_2 V_{t2} - r_1 V_{t1}) = \ \rho Q(r_2 V_{t2} - r_1 V_{t1})$$

$$T_{sh} = \rho * \pi * d_2 * b * V_{n2}(r_2 V_{t2} - r_1 V_{t1})$$
$$= 1000 * \pi * 0.3 * 0.02 * 6.58 * (0.15 * 8.67 - 0.0)$$
$$= 161\ N.m$$

(b)

$$P_w = 1000 * 0.124 * (22.78 * 8.67 - 11.39 * 0.0) = 24490\ W$$

$$P_w = \frac{P_w}{746} = \frac{24490}{746} \approx 33\ hp$$

$$\eta_m = \frac{P_w}{P_{bhp}} \quad \rightarrow P_{bhp} = \frac{P_w}{\eta_m} = \frac{33}{0.76} \approx 44\ hp$$

(c)

Head (H) as

$$H = \frac{P_w}{\rho g Q} = \frac{1}{g}(u_2 V_{t2} - u_1 V_{t1}) = \frac{1}{9.81}(22.78 * 8.67 - 0.0) = 20\ m$$

9.8 Pump Performance Curves.

Performance charts are always plotted for constant shaft-rotation speed ***N*** (rpm). The basic independent variable is taken to be discharge ***Q*** in (gal /min or m^3/s). The dependent variables or output, are taken to be head ***H*** for liquid (pressure rise ***Δp*** for gases), brake horsepower P_{php} and efficiency***η***. Fig.(9.11) shows typical performance curves for centrifugal pump.

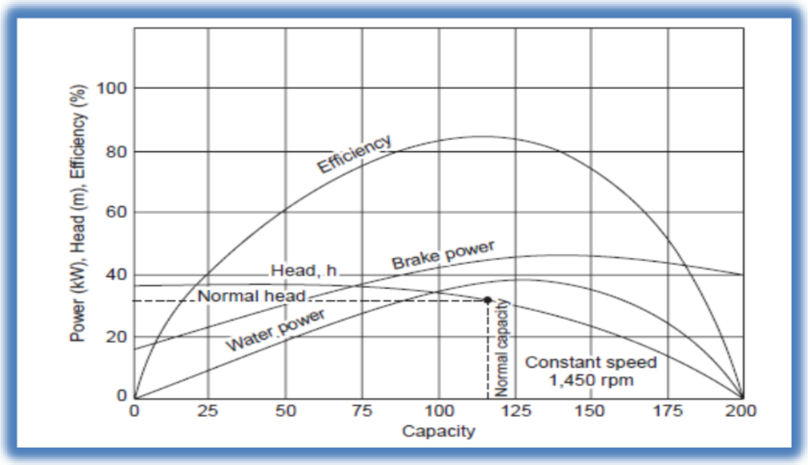


Figure (9.11) Centrifugal pump characteristics at constant speed.

The efficiency η is always zero at no flow and Q_{max}, and it reaches a maximum perhaps 80 to 90 percent at about $0.6Q_{max}$. This is the design point flow rate Q^* or best efficiency point (BEP) and η_{max}. . The head and horsepower at BEP will be termed H^* and P^*_{bhp}.

9.9 Net Positive – Suction Head (NPSH).

NPSH, which is the head required at the pump inlet to keep the liquid from cavitation or boiling. The *NPSH* is defined as

$$NPSH = \frac{p_s}{\rho g} + \frac{V_s^2}{2g} - \frac{p_v}{\rho g} \tag{9.24}$$

Where (p_s &V_s) are the pressure and velocity at the pump inlet and p_v is the vapor pressure of the liquid. The right–hand side is equal or greater than (*NPSH*) in the actual system to avoid cavitation. Applying the energy equation between sump surface and the pump suction level.

$$\frac{p_s}{\gamma} + \frac{V_s^2}{2g} + Z_s + h_{fs} = \frac{p_a}{\gamma} \tag{9.25}$$

From Eq's (9.24&9.25)

$$NPSH = \frac{p_a}{\gamma} - \frac{p_v}{\gamma} - Z_s - h_{fs} \tag{9.26}$$

Thoma cavitation parameter is defined by

$$\sigma = \frac{(NPSH)}{H} = \frac{\left(p_a/\gamma\right) - \left(p_v/\gamma\right) - Z_s - h_{fs}}{H}$$

At cavitation conditions

$$\sigma = \sigma_c \ and \ \frac{p_s}{\gamma} = \frac{p_v}{\gamma}$$

$$\sigma_c = \frac{\left(p_a/\gamma\right) - \left(p_v/\gamma\right) - Z_s - h_{fs}}{H} = \frac{V_s^2}{2gH} \tag{9.27}$$

The height of suction, the frictional losses in the suction line play an important role for avoiding cavitation at a location.

9.10 Outlet Blade Angle and Specific Speed.

Different blade arrangements can be used in design of centrifugal pumps. Three possible orientations of outlet blade angle. Forward curved blade (β_2 >90º), radial curved blade (β_2=90º) and backward curved blade (β_2 < 90º). Fig. (9.12) are shown the velocity triangles for three arrangements.

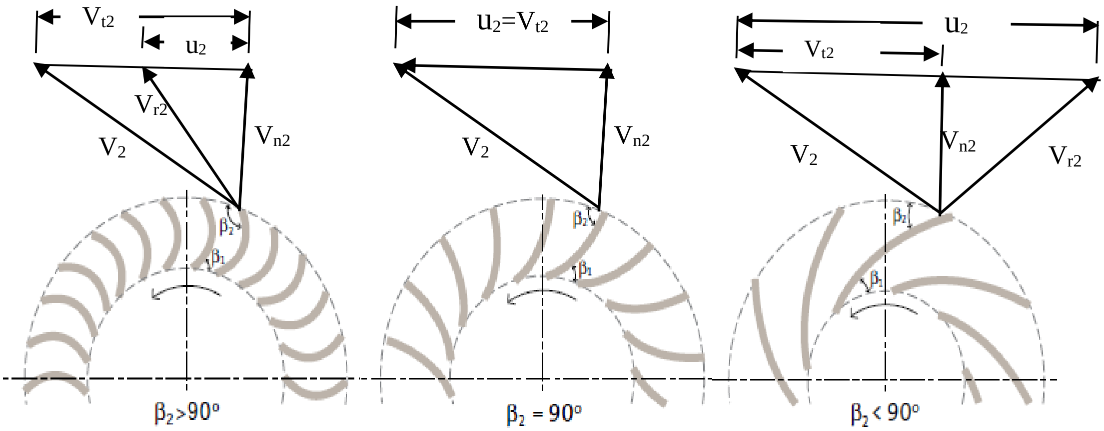


Figure (9.12) Blade shape arrangements with outlet velocity triangles

The dimensionless parameter (N_s) is known as the specific speed of pumps. In practice is used as

$$N_s = \frac{N\sqrt{Q}}{H^{3/4}} \tag{9.28}$$

The (N_s) quantity have been derived from dimensional analysis technique, and defined as the speed at which the pump will operate to deliver unit flow under unit head.

Ex.3

The outer diameter and width of a centrifugal pump impeller are 55 cm and 3 cm. The pump runs at 1300 rpm. The suction head is 7 m and the delivery head is 45 m. The frictional drops in suction side is 2.5 m and in the delivery side is 9 m. The blade angle at out let is 33°. The manometric efficiency is 83% and the overall efficiency is 77%. Determine

a- The power required to drive the pump.

b- Calculate the pressures at the suction and delivery side of the pump.

Sol.

Assume the inlet tangential velocity is zero, the total head against the pump is

H = 45+7+2.5+9= 63.5 m

$$u_2 = \frac{\pi d_2 N}{60} = \frac{\pi * 0.55 * 1300}{60} = 37.4\,\frac{m}{s} \qquad ; \qquad since \quad u_2 = \omega r_2$$

$$\eta_m = \frac{gH}{u_2 V_{t2}} = 0.83 = \frac{9.81 * 63.5}{37.4 * V_{t2}} \qquad solving\ for\ V_{t2} = 20.0\ m/s$$

The outlet velocity triangle is used to calculate V_{n2},

$tan\beta_2 = \frac{V_{n2}}{u_2 - V_{t2}}$

$\therefore\ V_{n2} = tan\beta_2(u_2 - V_{t2}) = tan33(37.4 - 20.0) = 11.36\,m/s$

Flow rate Q is

$$Q = \pi * d_2 * b_2 * V_{n2} = \pi * 0.55 * 0.03 * 11.36 = 0.58886\,m^3/s$$

$$\eta_o = \frac{P_w}{P_{bhp}} \quad Solving\ for\ \ Power\ (P_{bhp}) = \frac{P_w}{\eta_o} = \frac{\rho g Q H}{\eta_o}$$

$$\therefore\ P_{bhp} = \frac{1000 * 9.81 * 0.58886 * 63.5}{0.77} = 476391.56\,W$$

Now, consider the water level and the suction level are (1 & 2)

$\frac{p_a}{\gamma} + \frac{{V_1}^2}{2g} + Z_1 = \frac{p_2}{\gamma} + \frac{{V_2}^2}{2g} + Z_2 + losses$

$\frac{101000}{9810} + 0 + 0 = \frac{p_2}{9810} + \frac{11.36^2}{2 * 9.81} + 7 + 2.5 \quad solving\ for\ {}^{p_2}/_{\gamma}$

${}^{p_2}/_{\gamma} = 1.218\,m\ absolute$

Consider suction side and delivery side as 2 and 3,

$\frac{p_2}{\gamma} + \frac{{V_2}^2}{2g} + H_{ideal} = \frac{p_3}{\gamma} + \frac{{V_3}^2}{2g} + Z_3 \qquad ; since\ H_{ideal} = \frac{u_2 V_{t2}}{g}$

$V_3 = \sqrt{V_{t2}^2 + V_{n2}^2} = \sqrt{20^2 + 11.36^2} = 23\,m/s$

$$1.218 + \frac{11.36^2}{2 * 9.81} + \frac{37.4 * 20}{9.81} = \frac{p_3}{\gamma} + \frac{23^2}{2 * 9.81}; solving\ for$$

$$\frac{p_3}{\gamma} = 57\ m\ \ absolute.$$

Ex.4

A centrifugal pump was tested for cavitation initiation. Total head was 40 m and flow rate was 0.06 m3/s. Cavitation started when the total head at the suction side was 3 m. The atmospheric pressure was 760 mm Hg and the vapor pressure at this temperature was 2 kPa. It was proposed to install the pump where the atmospheric pressure is 700 mm Hg and the vapor pressure at the location temperature is 1 kPa. If the pump develops the same total head and flow, can the pump be fixed as the same height as the lab setup? **What should be the new height?**

Sol.

At the suction point

Total head = Vapour pressure + velocity head.

∴ Velocity head = Total head – Vapor pressure in head of water

$\frac{V_s^2}{2g} = 3 - \frac{2*10^3}{10^3*9.81} = 2.796\,m$

Cavitation parameter (σ) is defined by

$\sigma = \frac{V_s^2}{2*g*H} = \frac{2.796}{40} = 0.0699$

Applying B.E. between water level and suction point,

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{p_{atm}}{\gamma}-h_f$$

$$\left(Z_1+h_f\right)=\frac{p_{atm}}{\gamma}-\sigma H-\frac{p_v}{\gamma}=\left(\frac{760*13.6}{1000}\right)-2.796-\frac{2*10^3}{10^3*9.81}=7.336\,m$$

At the new location (head and flow rate being the same, friction loss will be the same)

$$\left(Z_1{}'+h_f{}'\right)=\frac{p_{atm}}{\gamma}-\sigma H-\frac{p_v}{\gamma}=\left(\frac{700*13.6}{1000}\right)-2.796-\frac{1*10^3}{10^3*9.81}=6.622\,m$$

$$h_f=h_f'$$

$$\therefore\ (Z_1-Z_1')=0.714\,m$$

The pump should be lowered by 0.714 m, since the new height is 6.622 m.

Problems.

P9.1 Determine the discharge flow rate from centrifugal pump at 1000 rev./min, the head being 14.5 m. the vane angle at outlet is 30º to the periphery. The impeller diameter is 0.3 m and width is 0.05 m. The manometric efficiency of the pump is 0.85%. ***[1.14 m³/s]***

P9.2 A centrifugal pump impeller is 0.5 min diameter and delivers 2 m^3/min of water. The peripheral velocity is 10 m/s and the flow velocity is 2 m/s. the blade outlet angle is 35º. Whirl at inlet is zero. Determine the power and torque delivered by the impeller. ***[2.18 kW, 54.5 N m]***

P9.3 A centrifugal pump with 2.3 m diameter impeller running at 327 rev./min. delivers 7.9 m^3/s of water. The head developed is 72.8 m. The width of the impeller at outlet is 0.22 m. if the overall efficiency is 91.7% determine the power to drive the pump. Also determine the blade angle at exit. ***[61.52 kW, 13º]***

P9.4 A centrifugal pump running at 1000 rev./min. works against a head of 80 m delivering 1 m^3/s. The impeller diameter and width are 80 cm and 8 cm respectively. Leakage loss 3 percent of discharge. Hydraulic efficiency is 80%. External mechanical loss is 10kW. Calculate the blade angle at outlet, the power required and the overall efficiency. ***[15.5º, 1020kW, 76.9%]***

P9.5 The diameters of a centrifugal pump impeller is 600 mm and that of the eye is 300 mm. The vane angle at inlet is 30° and that at outlet is 45°. If the absolute velocity of water at inlet is 2.5 m/s determine the speed and manometric head. The whirl at inlet is zero. ***[275.8 rpm, 5.44 m]***

P9.6 The speed of a centrifugal pump was 240 rpm and it is required to develop 22.5 m head when discharging 2 m3/s of water. The flow velocity at outlet is 2.5 m/s. The vanes at outlet are set back at 30° to the tangential direction. Determine the manometric efficiency and the power required to drive the pump. Impeller diameter is 1.5 m. ***[81%, 545 kW]***

P9.7 A centrifugal pump delivers 50 l/s when running at 1500 rpm. The inner and outer diameters are 0.15 m and 0.25 m respectively. The blades are curved at 30° to the tangent at the outlet. The flow velocity is 2.5 m/s and is constant. The suction and delivery pipe diameters are 15 cm and 10 cm, respectively. The pressure head at suction is 4 m below atmosphere. The pressure at the delivery is 18 m above atmosphere. The power required was 18 kW. Determine the vane angle at inlet for zero whirl at inlet. Also find the manometric efficiency and overall efficiency.

[12°, 77.2%, 64.6%]

P9.8 A centrifugal pump running at 1450 rpm delivers 0.11 m3/s of water against a head of 23 m. The impeller diameter is 250 mm and the width is 50 mm. The manometric efficiency is 75% determine vane angle at outlet.

[42.1°]

P9.9 A six stages centrifugal pump running at 660 rpm is to deliver 0.8 m^3/s of water of water against a manometric head of 88 m. The vanes are curved back at 38° to the tangent at outer periphery. The velocity of flow is 35% of the peripheral velocity at outlet. The hydraulic losses are 33% of the velocity head at the outlet of the impeller. Determine the diameter of the impeller at outlet and the manometric efficiency. ***[0.5 m, 87.4%]***

P9.10 A multi stage pump is required to deliver of 2.5 l/s water against a maximum discharge head of 250 m. The diameter of radial bladed impeller should not be more than 18 cm. Assume a speed of 3000 rpm. Determine the impeller diameter, number of stages and power. ***[160 mm, 4, 8.76 kW]***